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DYNAMIC ANALYSIS OF COOLANT CIRCULATION IN BOILING WATER NUCLEAR REACTORS

by

Chathilingath K. Sanathanan

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Reactor Physics Division

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NOMENCLATURE

$a_0, a_1, a_2, \ldots$	Coefficients of the series expansion for ΔV_f
A	Cross-sectional area of channel
A_d	Cross-sectional area of downcomer
dA	Infinitesimal portion of A
$A_0, A_1, A_2, \ldots$	Coefficients for the series expansion of V_{f0}
$b_0, b_1, b_2, \ldots$	Coefficients for the series expansion of $\Delta\alpha$
$B_0, B_1, B_2, \ldots$	Coefficients for the series expansion of α_0
$c_0, c_1, c_2, \ldots$	Coefficients for the series expansion of $f(x)$
C	Time-dependent part of $\Delta\phi$
e	Internal energy per unit mass of fluid
f	Axial profile of $\Delta\phi$
F	Certain function
g	Acceleration due to gravity
G	Mass flow rate per unit area
ΔG	Perturbation of G
h, H	Enthalpy of fluid per unit mass
ΔH	Perturbation in H
h_f	Enthalpy of saturated water per unit mass
h_g	Enthalpy of saturated steam per unit mass
K_a, K_{aD}	Coefficient to obtain acceleration pressure drop in channel and downcomer, respectively
K_f, K_{fD}	Coefficient to obtain friction pressure drop in channel and downcomer, respectively
ΔK_{fb}	Transient void reactivity feedback
$d\ell$	Infinitesimal WP
L	Channel length
L_b	Boiling length of the channel
L_D	Length of downcomer
M_1	Constant $= \rho_g h_g - \rho_f h_f$
M_4	Constant $= \rho_g h_g S_0 - \rho_f h_f$
n	Degree of the Legendre polynomials
N_1	Constant $= \rho_g - \rho_f$
N_4	Constant $= \rho_g S_0 - \rho_f$
$\vec{P}$	Pressure per unit area of v
P	State pressure
$P_0, P_1, P_2, \ldots$	Legendre polynomials
ΔP	Perturbed pressure drop along the channel
s	Complex variable due to Laplace transformation
S	Surface of v; slip ratio $= V_g/V_f$
$S_0, S_1, S_2, \ldots$	Coefficients for the series expansion of slip ratio S
dS	Infinitesimal surface of S
t	Independent time variable
v	Control volume (See Fig. 2)
dv	Infinitesimal volume in v
$\vec{V}$	Velocity of fluid
V	Magnitude of $\vec{V}$
V_f, V_g	Velocity of water and steam, respectively
V_z	z-component of $\vec{V}$
V_{f0}	Steady-state velocity of water
ΔV_f	Perturbation of V_f
WP	Wetted perimeter of control volume v
x	Independent variable of the Legendre polynomials
x_m	Location of the extremum of $\Delta\alpha$
z	Independent space variable; independent variable of the Legendre polynomials
Δz	Infinitesimal z; length of the control volume v
α	Void fraction
α_0	Steady state of α
$\Delta\alpha$	Perturbation of α
$\gamma_0, \gamma_1, \gamma_2, \ldots$	Coefficients of series expansion of $\eta(x)$
Δ, d	To denote perturbation or infinitesimal quantity
$\vec{\zeta}$	Body force per unit mass
η	Reactivity worth of void
ρ	Density of fluid
ρ_f, ρ_g	Density of water and steam, respectively
$\bar{\rho}$	Average of ρ at any z
ρ'	Effective density for momentum considerations
ρ''	Effective slip flow density for energy considerations
σ	Real part of s
ω	Imaginary part of s
τ	Component in the negative z direction of wall friction force per unit perimeter per unit axial length
ϕ	Heat flux per unit length of channel
ϕ_0	Steady state of ϕ
$\Delta\phi$	Perturbation of ϕ
ψ	Heat flux per unit area of the surface of control volume v

DYNAMICS ANALYSIS OF COOLANT CIRCULATION IN BOILING WATER NUCLEAR REACTORS

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ABSTRACT

The dynamics of two-phase flow through the coolant channels of a natural-circulation boiling water nuclear reactor is studied analytically. One-dimensional conservation equations describing the flow through each channel are written in the linearized perturbed form, and Laplace transformation in time is performed. A systematic procedure is developed to approximate the solution. The solution may be oscillatory both in time and space, and the stability depends largely upon the steady-state profile of velocity and void fraction along the channel, as well as the channel length. The simplifying assumption made by earlier investigators that the slip ratio is constant along the channel length is shown to yield results close to the true solution.

The solution gives the space-time dependent transient steam void fraction which when multiplied by the reactivity worth of void yields the space-time dependent void reactivity. Coupling between parallel channels due to a common downcomer is investigated.

Several specific problems are considered and the predicted solutions are substantiated by comparison with those obtained through elaborate numerical methods and previous observations.

The analytical techniques developed are applicable to both natural- and forced-circulation systems.

I. INTRODUCTION

"I have no doubt whatever that our ultimate aim must be to describe the sensible in terms of the sensible."

J. H. Poynting

A good understanding of the dynamics of two-phase flow through heated boiling channels is essential to the design of high-performance boilers and boiling nuclear reactors. Because of the effect of steam voids on the reactivity, and hence on the control of a nuclear reactor, it is particularly important to understand the transient behavior of steam voids in the coolant channels.

Several attempts(1-5) have been made in the past years to uncover the possibility of instability, flow oscillations, or chugging in heated boiling channels with forced or natural circulation. The philosophy of almost all the past theoretical approaches has been to arrive at a solution to the equations expressing the conservation of energy, mass, and momentum pertinent to the system on the basis of several simplifying assumptions. In some of the elaborate numerical methods introduced by Meyer et al.,(2,4) the simplifying assumptions have been kept to a minimum, and there has been good agreement between the predicted and experimentally observed results.

There are two main disadvantages to the numerical approach, however, namely, that tremendous care must be exercised with respect to the computer programming each time the method is applied to different systems, and, secondly, that an explicit functional form of the time-space dependence of the solution is not obtained directly. Also, because of this lack of functional form, the numerical approach does not readily lend itself to analysis of complementary problems.

A second approach, the transfer-function technique, was proposed by Quandt(1) to study the response of two-phase flow to a given change in the heat flux. In this technique the space-time-dependent conservation equations were solved in their linearized perturbed form, there being assumed a knowledge of the steady state and one-dimensional space.

In Quandt's method, however, one is expected to know the exact spatial dependence of the solution even before the equations are solved. Specifically, the method assumes that the perturbed mass flow rate is linear along the length of the channel and that the perturbed fluid enthalpy always follows the integrated perturbed heat flux. These assumptions may not introduce large errors in the predicted dynamic behavior if the channel transport time is small compared with the period of flow oscillation. This was the case in the particular experimental setup which was used to observe the oscillations, and satisfactory agreement between the theoretical predictions and experimental observations was found. One may observe that these assumptions force the perturbations to have their extremums at the channel end points at all times, which is not true.

The present study is an attempt to avoid the above difficulties in obtaining both the time and space variations of the solution of the linearized perturbed conservation equations. To be sure, assumptions such as the flow is separated, the variables are unidimensional in space, and so forth, are made in the derivation of these equations. These assumptions are given in Section II.

The following is a very brief introduction to the present method. Basically, two things are desired: (1) the space-time solution of the linearized perturbed conservation equations, which yield such things as the transient void reactivity in a nuclear reactor, and (2) an understanding of the hydrodynamic stability. Consider one such desired result, the dependent variable $\Delta\alpha(z,t)$, perturbed void fraction which is a function of time 't,' and position 'z' along the length of the channel. In the present method $\Delta\alpha(z,t)$ is expressed as follows:

$$\Delta\alpha(z,t) = b_0P_0(z) + b_1P_1(z) + b_2P_2(z) + \ldots,$$

where the coefficients b_0, b_1, b_2, and so forth, are unknown functions of time only, and P_0, P_1, P_2, and so forth, are orthogonal functions of variable 'z.' Similar expressions, of course, are used for the other dependent variables. The nature of the problem and the boundary conditions suggest the use of Legendre polynomials, as discussed later. The coefficients b_0, b_1, and so forth, are obtained by substituting the above series expansions for the unknown variables in the equations and applying orthogonality conditions. Considerable evidence is available for the convergence of the above series approximation of the solution.

The hydrodynamic stability is investigated through transfer functions such as that between $\Delta V_f(z,t)$, the perturbed velocity of the liquid, and $\Delta\phi(z,t)$ the perturbed heat flux. This is done conveniently by use of a Laplace transformation of the time variable.

The method developed is applied to a number of specific problems. Dependence of flow stability upon several factors such as the channel length, the inlet pressure drop, the downcomer drops, and the steady-state heat-flux profile along the channel, is investigated for a specific natural-circulation loop. The problems of operating channels in parallel with a common downcomer are also considered.

In particular, the void reactivity is investigated, since it plays an important role in the stability of a nuclear reactor. As the z-dependence of $\Delta\alpha$ is available in this development, one may evaluate the transient reactivity contribution of each channel if the void coefficient of reactivity, $\eta(z)$, is given. The total void reactivity of the reactor is obtained by adding the individual contributions of the channels. In many instances in the past,[3,6] the void reactivity of the reactor was calculated by simply estimating an average of $\Delta\alpha$ and the void coefficient of reactivity for the entire reactor. Both of these methods are presented in Section IV, and the deficiency of the other method is illustrated.

Results indicate that though the steam slip ratio is allowed to be variable along a channel, in contrast to many previous investigations,[1,3,7] little change occurs in the solution if the simplifying assumption of constant steam slip ratio is made. It is also found that the assumption of a simple z-dependence of the solution (i.e., one linear in z) can introduce sizeable errors. In addition, the method predicts the steady-state perturbation of void fraction, velocity of the water, and so forth, for a step increase in heat input to a natural-circulation loop. In general, remarkable agreement is found between the predictions made by the present method and experimental observations made thus far, as well as those by elaborate digital-computer solutions.

II. THE MODEL AND THE ASSUMPTIONS

Figure 1 illustrates the model under consideration, which consists of a vertical reactor core, with several coolant channels, immersed in water. A typical flow loop is indicated on the diagram.

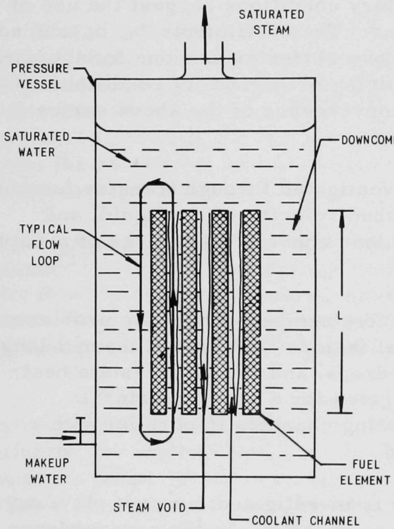

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Fig. 1. Model of a Natural-circulation Boiling Water Reactor

It is assumed that the reactor pressure and its power level may be varied, and that the water in the entire vessel and the makeup water are at saturation temperature. Further, there is no carryunder in the downcomer, and the pressure of the overall system does not change with time for small perturbations in power, so that the physical properties, such as the saturation temperature, do not change.

The occurrence of separated flow has been observed and is generally accepted, as it will be in this presentation. However, the lack of understanding of flow patterns makes it unrealistic to assign cross-sectional velocity distributions within the two phases. Furthermore, the fact that the cross section of the channels considered is not large compared with the bubble size, leads to the assignment of a single velocity $V_f(z,t)$ to the liquid particles and of $V_g(z,t)$ for the gas at any cross section located at a distance z from the inlet to the channel. A single variable $\alpha(z,t)$ is chosen to represent the fraction of the cross section at z which is occupied by steam. It is also assumed that the slip ratio $S = V_g/V_f$ does not change from its steady-state value for small perturbations in the heat input. Nevertheless, variation of the slip ratio along the channel length is considered. The value of S may, in general, be greater than or equal to one.

The statistical fluctuations in the heat flux, void fraction, velocity, etc., due to the randomness in bubble formation are assumed to be at much higher frequencies than the natural frequencies of oscillation in the system. Processes such as the eddy diffusion of energy and momentum are considered negligible, and so are the kinetic and potential energy terms, as well as the variation of pressure with time. Also, the terms representing the energy loss due to expansion at the exit of the channels along with the gain in pressure head are neglected. It is observed that this gain in pressure head and the loss due to the eddy diffusion of momentum in the riser tend to cancel each other in some steady-state experiments.(8) As there is a lack of fair understanding of the latter, it is only logical not to consider just the former alone.

Flow of heat from the wall surface to the fluid, taken to be without an internal heat source, is assumed normal to the wall surface. Perturbation of the heat flux is assumed to take place simultaneously in all channels. Further, in each channel the perturbed heat flux of the channel may be expressed as

$$\Delta\phi(z,t) = f(z)\ C(t), \qquad t > 0$$

and

$$\Delta\phi(z,o) = 0,$$

where f(z) is known and does not change with time, and C(t) is arbitrary but Laplace transformable. However, f(z), the z-dependent factor of $\Delta\phi$, is allowed to be different from $\phi_0(z)$, the steady-state heat flux for that channel. This fact may be utilized with some advantage in an experimental study.

The steady-state distributions of heat flux, velocity of the liquid, slip ratio, and void fraction along the channel are assumed to be known through experimental observations or through some of the existing semi-empirical computations. Surface frictional stresses are also considered to be obtainable empirically.

All the variables considered, such as the void fraction, and velocity of the liquid, are continuous and differentiable in time and space.

The origin of the z-axis is at the bottom end of the channel, and z is positive upwards.

III. FUNDAMENTAL CONSERVATION EQUATIONS*

The system to be considered is shown in Fig. 2.

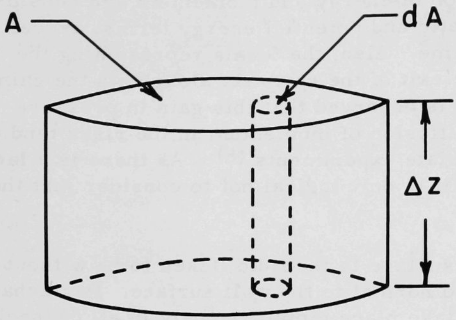

Fig. 2
Control Volume

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A. Conservation of Mass

Consider an arbitrarily small length Δz of the channel located z units above the inlet. Let A be the constant area of cross section of the channel, and v the volume corresponding to the length Δz; let the surface area be denoted by S. Let ρ be the density at any point in v, and $\vec{V}$ the velocity of the particle at that point.

Conservation of mass may be stated as,

$$\frac{\partial}{\partial t}\left[\int_v \rho \, dv\right] + \int_S \rho\vec{V} \cdot d\vec{S} = 0. \tag{1}$$

But

$$\int_v \rho \, dv = \int_z^{z+\Delta z}\left(\int_A \rho \, dA\right) dz. \tag{2}$$

Since there is no contribution from the zero velocity at the side walls,

$$\int_S \rho\vec{V} \cdot d\vec{S} = \left(\int_A \rho V_z \, dA\right)_{z+\Delta z} - \left(\int_A \rho V_z \, dA\right)_z , \tag{3}$$

where V_z is the component of $\vec{V}$ in the positive z-direction.

Equation (1) may be written, after dividing throughout by Δz and taking the limit as Δz approaches zero, as

*Derivation of these equations is similar to that by J. E. Meyer.[9]

$$\frac{\partial}{\partial t}\left(\int_A \rho \, dA\right) + \frac{\partial}{\partial z}\left(\int_A \rho V_z \, dA\right) = 0. \tag{4}$$

The assumption of separated flow leads to the following equations:

$$\int_A \rho \, dA = \left[\rho_f(1-\alpha) + \rho_g\alpha\right]A; \tag{5}$$

$$\int_A \rho V_z \, dA = \left[\rho_f V_f(1-\alpha) + \rho_g V_g\alpha\right]A, \tag{6}$$

where α is the void fraction, and the subscripts f and g refer to water and steam, respectively. Substituting the expressions for $\int_A \rho \, dA$ and $\int_A \rho V_z \, dA$ from Eqs. (5) and (6) into Eq. (4) and dividing by A, one may write

$$\frac{\partial}{\partial t}\left[\rho_f(1-\alpha) + \rho_g\alpha\right] + \frac{\partial}{\partial z}\left[\rho_f V_f(1-\alpha) + \rho_g V_g\alpha\right] = 0. \tag{7}$$

B. Conservation of Energy

The conservation of energy may be stated mathematically for the same volume v with no internal heat generation as follows:

$$\frac{\partial}{\partial t}\left[\int_v \rho e \, dv\right] + \int_S \rho e \, \vec{V} \cdot d\vec{S} = -\int_S \vec{\psi} \cdot d\vec{S}$$

$$+ \int_S (\vec{V} \cdot \vec{p}) dS + \int_v \rho\vec{V} \cdot \vec{\zeta} \, dv, \tag{8}$$

where e, the internal energy per unit mass, is given by

$$e = h - (P/\rho) + \frac{1}{2} V^2, \tag{9}$$

where h is the enthalpy, P the static pressure, ρ the density, V the magnitude of $\vec{V}$ at any point in the control volume, $\vec{\psi}$ the surface heat-flux vector, $\vec{p}$ the force per unit area on the surface of the control volume v, and $\vec{\zeta}$ the body force vector per unit mass.

If ϕ is the heat flux per unit length of the channel and g the acceleration due to gravity, Eq. (8), after performing operations similar to those leading to Eq. (4), becomes

$$\frac{\partial}{\partial t}\left(\int_A \rho e \, dA\right) + \frac{\partial}{\partial z}\left(\int_A \rho V_z e \, dA\right) = - \frac{\partial}{\partial z}\left(\int_A PV_z \, dA\right)$$

$$- g\left(\int_A \rho V_z \, dA\right) + \phi. \tag{10}$$

The assumptions made in Eq. (10) are that the heat conduction in the positive z-direction and the energy contribution from the pressure and shear forces operating on the side wall are negligible. Further, only the earth's gravity contributes to the body forces.

Substitution for e from Eq. (9) into Eq. (10) leads to

$$\frac{\partial}{\partial t}\left(\int_A \rho h \, dA\right) + \frac{\partial}{\partial z}\left(\int_A \rho V_z h \, dA\right) = \phi$$

$$+ \left[\frac{\partial}{\partial t}(PA) - \frac{\partial}{\partial t}\left(\int_A \frac{1}{2}\rho V^2 \, dA\right) + \frac{\partial}{\partial z}\left(\int_A \frac{1}{2}\rho V_z V^2 \, dA\right) + g\int_A \rho V_z \, dA\right]. \tag{11}$$

Under the assumptions that the pressure does not vary with respect to time and that the kinetic and potential energy terms have negligible effects, the expression in brackets on the right-hand side of Eq. (11) may be neglected. Thus,

$$\frac{\partial}{\partial t}\left(\int_A \rho h \, dA\right) + \frac{\partial}{\partial z}\left(\int_A \rho V_z h \, dA\right) = \phi. \tag{12}$$

Using the assumption of separated flow, one may write

$$\int_A \rho h \, dA = A\left[\rho_f h_f(1 - \alpha) + \rho_g h_g \alpha\right] \tag{13}$$

and

$$\int_A \rho V_z h \, dA = A\left[\rho_f h_f V_f(1-\alpha) + \rho_g h_g V_g \alpha\right]. \tag{14}$$

Substitution of Eqs. (13) and (14) into Eq. (12) and division throughout by A yields

$$\frac{\partial}{\partial t}\left[\rho_f h_f(1-\alpha) + \rho_g h_g \alpha\right] + \frac{\partial}{\partial z}\left[\rho_f h_f V_f(1-\alpha) + \rho_g h_g V_g \alpha\right] = \phi/A. \tag{15}$$

C. Conservation of Momentum

The equation for momentum balance for the length Δz is written to express the incremental pressure drop for this length. Integration along the entire channel would yield the channel pressure drop to which the entry and exit drops may be added. The resulting expression may be a constant in a forced-circulation system (see Ref. 1). In a natural-circulation system, the pressure drop around a closed circulation loop (a typical loop is indicated in Fig. 1) is equated to zero. This is one of the boundary conditions.

The momentum balance for the control volume v may be stated as

$$\frac{\partial}{\partial t}\int_v \rho\vec{V} \, dv + \int_S (\rho\vec{V})\,\vec{V}\cdot d\vec{S} = \int_S \vec{p}\, dS + \int \rho\vec{\zeta}\, dv. \tag{16}$$

The pressure drop is considered constant across any cross section normal to the z-axis, that is, $P = P(z)$. Hence, the first term on the right-hand side of Eq. (16) may be written as

$$\int_S \vec{p}\, dS = -(PA)_{z+\Delta z} + (PA)_z + \int_z^{z+\Delta z}\left(P\frac{dA}{dz}\right)dz - \int_z^{z+\Delta z}\left(\int_{WP} \tau \, d\ell\right)dz, \tag{17}$$

where WP is the total and $d\ell$ the incremental wetted perimeter, and τ the frictional force per unit of the wetted surface. Since the channel has constant area of cross section, $dA/dz = 0$.

The second term on the right-hand side of Eq. (16) represents the body force per unit mass of the control volume. If it is assumed that this force is due only to the earth's gravity, one may write

$$\int_V \rho\vec{\zeta}\ dV = -\int_z^{z+\Delta z}\left(\int_A \rho g\ dA\right)dz. \tag{18}$$

Hence, the conservation of momentum expressed by Eq. (16), after making substitutions through use of Eqs. (17) and (18), dividing throughout by Δz, and taking the limit as Δz tends to zero, may be restated as

$$\frac{\partial}{\partial t}\left(\int_A \rho V_z\ dA\right) + \frac{\partial}{\partial z}\left(\int_A \rho V_z^2\ dA\right) = -A\frac{\partial P}{\partial z} - \int_{WP} \tau\ d\ell - g\left(\int_A \rho\ dA\right). \tag{19}$$

The pressure drop per unit length due to surface friction of the wall, $\int_{WP} \tau\ d\ell$, is empirically evaluated by assuming the drop to be entirely due to water. Specifically, it is assumed that

$$\frac{1}{A}\int_{WP} \tau\ d\ell = K_f V_f^2, \tag{20}$$

where K_f is empirically obtained. (See Ref. 7 for an experimental justification for this type of evaluation of the two-phase friction pressure drop.)

Again, the assumption of separated flow enables one to write:

$$\int_A \rho V_z\ dA = A\left[\rho_f V_f(1-\alpha) + \rho_g V_g \alpha\right]; \tag{21}$$

$$\int_A \rho V_z^2\ dA = A\left[\rho_f V_f^2(1-\alpha) + \rho_g V_g^2 \alpha\right]; \tag{22}$$

$$\int_A \rho\ dA = A\left[\rho_f(1-\alpha) + \rho_g \alpha\right]. \tag{23}$$

Upon substitution of Eqs. (20), (21), (22), and (23) into Eq. (19), and dividing throughout by A, one obtains

$$-\frac{\partial P}{\partial z} = K_f V_f^2 + \left[\rho_f(1-\alpha) + \rho_g\alpha\right]g + \frac{\partial}{\partial t}\left[\rho_f V_f(1-\alpha) + \rho_g V_g \alpha\right] + \frac{\partial}{\partial z}\left[\rho_f V_f^2(1-\alpha) + \rho_g V_g^2\alpha\right]. \tag{24}$$

Equations (7), (15), and (24) give the nonstationary equations of conservation of mass, energy, and momentum in one-dimensional space.

IV. DEVELOPMENT OF AN APPROXIMATION PROCESS FOR THE SOLUTION OF THE CONSERVATION EQUATIONS WRITTEN IN THE LINEARIZED PERTURBED FORM

The response of the two-phase flow to small changes in the heat flux may be obtained from the solution of the linearized perturbed form of the conservation equations about a steady operating point. With this information, one may also predict the stability of the system for small disturbances. By utilizing the technique of small perturbations about some steady-state operating condition, the variables may be expressed as follows:

$$\phi(z,t) = \phi_0(z) + \Delta\phi(z,t); \tag{25}$$

$$V_f(z,t) = V_{f0}(z) + \Delta V_f(z,t); \tag{26}$$

$$\alpha(z,t) = \alpha_0(z) + \Delta\alpha(z,t), \tag{27}$$

where the first term on the right-hand side of each equation is the steady-state term and the second term is the perturbation in the corresponding variable.

Let it be assumed that the velocity of steam may be expressed as

$$V_g(z,t) = S(z)V_f(z,t), \tag{28}$$

where $S(z)$ is the slip ratio along the channel and is assumed not to be a function of time for small perturbations in power. The value of $S(z)$ is assumed to be available from steady-state information.

A. Linearized Perturbed Form of the Conservation Equations

By substituting Eqs. (25), (26), (27), and (28) into the conservation Eqs. (7), (15), and (24), performing a Laplace transformation in time, and eliminating the steady-state, second-, and higher-order terms, the following is obtained:

$$s\left(\rho_g - \rho_f\right)\Delta\alpha + \left(\rho_g S - \rho_f\right)\frac{\partial}{\partial z}\left[\alpha_0 \Delta V_f + V_{f0}\Delta\alpha\right] + \rho_f \frac{\partial \Delta V_f}{\partial z}$$

$$+ \rho_f \frac{\partial \Delta V_f}{\partial z} + \frac{dS}{dz}\rho_g\left[\alpha_0 \Delta V_f + V_{f0}\Delta\alpha\right] = 0; \tag{29}$$

$$s\left(\rho_g h_g - \rho_f h_f\right)\Delta\alpha + \left(S\rho_g h_g - \rho_f h_f\right)\frac{\partial}{\partial z}\left[\alpha_0 \Delta V_f + V_{f0}\alpha\right]$$

$$+ \rho_f h_f \frac{\partial \Delta V_f}{\partial z} + \frac{dS}{dz}\rho_g h_g \left[\alpha_0 \Delta V_f + V_{f0}\Delta\alpha\right] = \frac{\Delta\phi}{A}; \tag{30}$$

$$- \frac{\partial \Delta P}{\partial z} = 2K_f V_{f0}\Delta V_f + g\left(\rho_g - \rho_f\right)\Delta\alpha$$

$$+ s\left[\Delta\alpha\left(S\rho_g V_{f0} - \rho_f V_{f0}\right) + \Delta V_f\left(S\rho_g \alpha_0 - \rho_f \alpha_0\right) + \rho_f \Delta V_f\right]$$

$$+ \frac{\partial}{\partial z}\left[\Delta\alpha\left(S^2 \rho_g V_{f0}^2 - \rho_f V_{f0}^2\right)\right.$$

$$\left. + 2\Delta V_f\left(S^2\rho_g V_{f0}\alpha_0 - \rho_f V_{f0}\alpha_0\right) + 2\rho_f V_{f0}\Delta V_f\right]. \tag{31}$$

In these equations s is the Laplace transform variable, and $\Delta\alpha$, ΔV_f, and $\Delta\phi$ are functions of z and s. Equations (29), (30), and (31) represent the linearized perturbed form of the conservation equations. It is assumed that $\Delta\phi$ may be expressed in the following form:

$$\Delta\phi = f(z)C(s), \tag{32}$$

where $f(z)$ is continuous and differentiable and is known, and $C(s)$ is the Laplace transform of $C(t)$.

The perturbation in ϕ is assumed to occur at time $t = 0$.

Equation (31) may be integrated along the channel to obtain the perturbed pressure drop in the channel to which the perturbed pressure drop at the inlet, exit, and the downcomer may be added to obtain the perturbed pressure drop around a closed loop which is equated to zero. This is one of the boundary conditions. The second boundary condition is that $\Delta\alpha$ is equal to zero at $z = 0$ for all time t, because the inlet water is at saturation temperature and heat addition begins only from the inlet.

The problem now is to solve Eqs. (29) and (30) with the above boundary conditions to obtain $\Delta\alpha$ and ΔV_f as functions of both z and t for any arbitrary $C(t)$.

The procedure which shall be followed is to expand ΔV_f and $\Delta\alpha$ in a series of orthogonal functions of the independent space variable with the coefficients as unknown functions of time. These coefficients are then evaluated by the application of the orthogonality conditions and the existing boundary conditions. Although many functions are available, the Legendre polynomials appear to be the most convenient for several reasons.

First, a linear transformation of the space variable from z to x is desired, where $0 \leq z \leq L$ and $x_1 \leq x \leq x_2$; the interval (x_1, x_2) is the region of orthogonality of the function chosen with a simple weighting function if possible. This facilitates the application of the orthogonality conditions and the partial differentiation with respect to z in the conservation equations. Therefore, functions such as Laguerre polynomials would not be ideal.

Secondly, since α and V_f are quite smooth along the channel, the use of Fourier Series expansion may not be efficient. Moreover, the location of the extremum of α or V_f along the channel will involve the solution of transcendental equations, which is not a desirable feature.

Considering the above factors, two sets of polynomials, the Legendre and Radau, appear promising. The Radau polynomials, however, would give difficulty since ΔV_f is not equal to zero at the inlet whereas $\Delta\alpha$ is. But for this fact, the Radau polynomials might be the most efficient. One is led, therefore, to consider the Legendre polynomials as the most desirable. It should be noted that there is an advantage with the Legendre polynomials in that the weighting function introduced for the orthogonality conditions is unity.

The following linear transformation in z is made to obtain a variable x varying from -1 to +1 in order to apply the orthogonality conditions:

$$x = \frac{2z}{L} - 1. \tag{33}$$

Then ΔV_f and $\Delta\alpha$ are expressed as follows:

$$\Delta V_f = a_0 + a_1 P_1(x) + a_2 P_2(x) + \ldots; \tag{34}$$

$$\Delta\alpha = b_0 + b_1 P_1(x) + b_2 P_2(x) + \ldots . \tag{35}$$

With the change of variable z to x according to Eq. (33), Eqs. (29), (30), and (31) are modified as follows:

$$s \frac{L}{2} \left(\rho_g - \rho_f\right)\Delta\alpha + \left(\rho_g S - \rho_f\right) \frac{\partial}{\partial x} \left[\alpha_0 \Delta V_f + V_{f0}\Delta\alpha\right]$$

$$+ \rho_f \frac{\partial \Delta V_f}{\partial x} + \frac{dS}{dx} \rho_g \left[V_{f0}\Delta\alpha + \alpha_0 \Delta V_f\right] = 0; \tag{36}$$

$$s\frac{L}{2}\left(\rho_g h_g - \rho_f h_f\right)\Delta\alpha + \left(\rho_g h_g S - \rho_f h_f\right)\frac{\partial}{\partial x}\left[\alpha_0 \Delta V_f + V_{f0}\Delta\alpha\right]$$

$$+ \rho_f h_f \frac{\partial \Delta V_f}{\partial x} + \frac{dS}{dx}\rho_g h_g\left[V_{f0}\Delta\alpha + \alpha_0 \Delta V_f\right] = \frac{L}{2A} f(x)C(s); \qquad (37)$$

$$-\frac{\partial \Delta P}{\partial x} = LK_f V_{f0}\Delta V_f + \frac{L}{2} g\left(\rho_g - \rho_f\right)\Delta\alpha$$

$$+ \frac{L}{2} s\left[\left(V_{f0}\Delta\alpha + \alpha_0 \Delta V_f\right)\left(\rho_g S - \rho_f\right) + \rho_f \Delta V_f\right]$$

$$+ \frac{\partial}{\partial x}\left[\Delta\alpha V_{f0}^2\left(\rho_g S^2 - \rho_f\right) + 2V_{f0}\alpha_0\left(\rho_g S^2 - \rho_f\right)\Delta V_f + 2\rho_f V_{f0}\Delta V_f\right]. \qquad (38)$$

The fact that $\Delta\alpha = 0$ at $x = -1$ is stated as follows:

$$b_0 - b_1 + b_2 - b_3 + \ldots = 0. \qquad (39)$$

The steady-state quantities are expressed as

$$V_{f0} = A_0 + A_1P_1(x) + A_2P_2(x) + \ldots; \qquad (40)$$

$$\alpha_0 = B_0 + B_1P_1(x) + B_2P_2(x) + \ldots; \qquad (41)$$

$$S = S_0 + S_1P_1(x) + S_2P_2(x) + \ldots, \qquad (42)$$

where the coefficients are obtained through a curve-fitting procedure applied to the steady-state spatial variations of V_{f0}, α_0 and S by means of the minimum-mean-square-error criterion.(10) The space-dependent part of $\Delta\phi$, namely, f(z), is assumed to be expressed in the form

$$f(z) = c_0 + c_1P_1(x) + c_2P_2(x) + \ldots, \qquad (43)$$

where c_0, c_1, etc., are known.

To obtain the transient solution desired, it is necessary to solve for $\Delta\alpha$ and ΔV_f. This is accomplished by an approximation procedure.

B. Procedure for the First-degree Approximation

A first-degree approximation in z, and hence in x, may be made for ΔV_f and $\Delta\alpha$ as follows:

$$\Delta V_f = a_0 + a_1 P_1(x); \tag{44}$$

$$\Delta \alpha = b_0 + b_1 P_1(x). \tag{45}$$

The expressions for the steady-state quantities, and for ΔV_f and $\Delta\alpha$ given by Eqs. (40) to (45) are substituted in Eqs. (36), (37), and (38). There are 5 unknowns, namely, a_0, a_1, b_0, b_1, and C. The two boundary conditions, and the integration of Eqs. (36) and (37) in x between the limits -1 and +1 after they have been multiplied by $P_0(x)$ (orthogonality conditions) give rise to 4 independent equations in the independent variable s. Hence, the ratios $a_0(s)/C(s)$, $a_1(s)/C(s)$, $b_0(s)/C(s)$, and $b_1(s)/C(s)$ may be readily evaluated as functions of s. These are called the transfer functions. If C(s) is specified, the s dependence and hence the time dependence of a_0, a_1, b_0, and b_1 may be obtained. It is to be noted here that C(s) becomes known when a particular perturbation in the heat flux, $\Delta\phi(z,t)$, is specified.

C. Procedure for the Second-degree Approximation

If a second-degree approximation in z is required, ΔV_f and $\Delta\alpha$ are expressed as

$$\Delta V_f = a_0 + a_1 P_1(x) + a_2 P_2(x) \tag{46}$$

and

$$\Delta \alpha = b_0 + b_1 P_1(x) + b_2 P_2(x). \tag{47}$$

It is readily observed that there are 7 unknowns. The boundary conditions yield 2 independent equations. Further, each of the Eqs. (36) and (37) are multiplied by $P_0(x)$ and $P_1(x)$, and integrated over x between -1 and +1 (orthogonality conditions) to generate 4 more independent equations. Therefore, the ratios a_0/C, a_1/C, a_2/C, b_0/C, b_1/C, and b_2/C may be evaluated.

One may thus readily proceed to approximations of higher degree for ΔV_f and $\Delta\alpha$ in the above systematic manner. A detailed illustration of this approximation procedure is given in Appendix A.

At this point, two very important results which follow directly from the present method are obtained.

D. Oscillatory Nature of the Solution for $\Delta\alpha$ in Space

From the solution $\Delta\alpha(x,s)$ one may locate the point $x_m(s)$ at which $\Delta\alpha(x,s)$ is an extremum. This may be demonstrated for a second-degree approximation as follows. From Eq. (47) we find that

$$\frac{\partial \Delta\alpha(x,s)}{\partial x} = b_1 \frac{d}{dx} P_1(x) + b_2 \frac{d}{dx} P_2(x)$$

$$= b_1 \frac{d}{dx}(x) + b_2 \frac{d}{dx}\left[\frac{1}{2}(3x^2 - 1)\right]$$

$$= b_1 + 3 b_2 x. \tag{48}$$

To find the location of the extremum of $\Delta\alpha$ along the channel, one may equate $\partial\Delta\alpha/\partial x$ to zero; we thereby obtain

$$x_m(s) = -b_1(s)/3 b_2(s). \tag{49}$$

One may infer from Eq. (49) that the location of the extremum of the transient void fraction may, in general, vary during transients. Hence, it may be concluded that the spatial distribution of $\Delta\alpha$ also undergoes spatial oscillations. Such an oscillatory nature of $\Delta\alpha$ cannot be obtained by a linear spatial approximation of $\Delta\alpha$, but only by a second- or higher-degree approximation. The need for the present method thus becomes evident.

E. Void Reactivity Feedback

In a boiling water reactor, reactivity feedback occurs due to the formation of steam bubbles in the coolant channels. In a transfer-function analysis of the system, knowledge about the feedback transfer function resulting from the transient void formation in the channels is necessary.

The present approximation procedure enables one to obtain $\Delta\alpha(z,s)$ given $\Delta\phi(z,s)$ in any channel.

Given $\eta(z)$, the reactivity worth of void along the channel, one may evaluate the reactivity feedback $\Delta K_{fb}(s)$ due to that channel as follows:

$$\Delta K_{fb}(s) = \int_0^L \Delta\alpha(z,s)\,\eta(z)\,dz. \tag{50}$$

This kind of evaluation was not possible in the past approaches except numerically, because $\Delta\alpha(z,s)$ was not known analytically. In many instances in the past, $\Delta K_{fb}(s)$ was computed from the knowledge of $\overline{\Delta\alpha}(s)$, the average of $\Delta\alpha(z,s)$, and $\bar{\eta}$, the average of $\eta(z)$ along z as

$$\Delta K_{fb}(s) = L\bar{\eta}\overline{\Delta\alpha}(s). \tag{51}$$

The following example illustrates the inadequacy of the previous method. The error introduced in $\Delta K_{fb}(s)$ by the previous method would, of course, vary with the z-dependence of $\Delta\alpha(z,t)$ and $\eta(z)$. Using the transformation of z to x defined by Eq. (33), and expanding $\Delta\alpha(z,t)$ and $\eta(z)$ in a series of Legendre polynomials, we obtain

$$\eta(x) = \gamma_0 + \gamma_1 P_1(x) + \gamma_2 P_2(x) + \ldots, \tag{52}$$

where γ_0, γ_1, γ_2, etc., are known constants and

$$\Delta\alpha(x,s) = b_0(s) + b_1(s)\, P_1(x) + b_2(s)\, P_2(x) + \ldots. \tag{53}$$

Thus $\overline{\Delta\alpha}(s) = b_0(s)$; and $\overline{\eta} = \gamma_0$.

Using the present method, we find

$$\begin{aligned}\Delta K_{fb}(s) &= \int_{-1}^{+1} \Delta\alpha(x,s)\, \eta(x)\, \frac{L}{2}\, dx \\ &= \frac{L}{2} \int_{-1}^{+1} \left[b_0 + b_1 P_1(x) + b_2 P_2(x) + \ldots\right] \\ &\quad \left[\gamma_0 + \gamma_1 P_1(x) + \gamma_2 P_2(x) + \ldots\right] dx \\ &= L\left[b_0\gamma_0 + \frac{b_1\gamma_1}{3} + \frac{b_2\gamma_2}{5} + \ldots\right].\end{aligned} \tag{54}$$

According to the previous method,

$$\Delta K_{fb}(s) = L\overline{\Delta\alpha}(s)\overline{\eta} = L b_0 \gamma_0. \tag{55}$$

Comparison of Eqs. (54) and (55) illustrates the possibility of error in the estimation of $\Delta K_{fb}(s)$ by the previous method.

The transfer function $\Delta K_{fb}(s)/C(s)$ may be written by use of Eq. (54) as

$$\frac{\Delta K_{fb}(s)}{C(s)} = L\gamma_0 \frac{b_0(s)}{C(s)} + \frac{L\gamma_1}{3}\frac{b_1(s)}{C(s)} + \frac{L\gamma_2}{5}\frac{b_2(s)}{C(s)} + \ldots. \tag{56}$$

The right-hand side of Eq. (56) has a common denominator $|A(s)|$ according to Eq. (A-34a) of Appendix A. Therefore, $\Delta K_{fb}(s)/C(s)$ may be readily computed to obtain the feedback reactivity transfer function due to one channel. The other channels may also be treated similarly, and the results are added to obtain the overall reactivity feedback transfer function of the entire reactor. In many cases, it is enough to divide the core into 2 or 3 regions and assign just one $\Delta\,\phi(z,s)$ for each region. All the coolant channels in that region would then be associated with a single $\Delta\phi(z,s)$. In this way, when there are many coolant channels in a core, $\Delta K_{fb}(s)$ for all those many channels need not be calculated separately. The number of regions would, of course, depend upon the spatial distribution of heat flux and the required accuracy of the result.

It is assumed in the above that $C(s)$ is common to all the channels. In other words, the change in the heat flux occurs simultaneously in all channels. This is true only if the change in the heat flux corresponding to a change in the thermal neutron flux in the reactor is delayed equally for all channels. This delay is due to the time taken for the heat to flow through the fuel element, cladding, etc.

V. DEMONSTRATION OF CONVERGENCE

The method of series approximation developed in the previous section is useful only if the series converges fast enough for practical application. It is necessary, therefore, to demonstrate convergence, although an absolute proof of convergence is not available at the present time. One may note that it is necessary that the solution converge in both space and time.

The following is chosen as an example of the system (FPS system of units is used). A natural-circulation loop with a single-heated channel, and a downcomer is considered. The channel has a diameter of 1 in. and is 4 ft long. The downcomer has a diameter of 6 in. and is 4 ft long. The steady-state condition is specified by the conditions

$$V_{f0} = 5.65 + 2.25\, P_1(x);$$

$$\alpha_0 = 0.325 + 0.325\, P_1(x);$$

$$S = 1.8.$$

The above simple space dependence of the steady state is chosen to illustrate the fact that the z dependence of the transients may be very different from that of the steady state.

The perturbation in heat is described below. We take

$$f(z) = 1.$$

C(t) is a unit step function, so that

$$C(s) = 1/s.$$

A. Convergence of the Major Pole Locations

To demonstrate convergence in time, the transfer function $a_0(s)/C(s)$ is considered. Table I gives the pole locations for the various approximations of $a_0(s)/C(s)$. Table I clearly shows that the pole positions converge. The major poles have converged to approximately $-3.77 \pm j5.35$. The higher-order approximations introduce poles whose real parts are considerably more negative than that of the major poles. It is found that the residues of all the poles have the same order of magnitude. Therefore, the time response is essentially dictated by the major poles.

Table I

POLE LOCATIONS

Degree of Approximation	Number of Poles	Locations of Poles: $\sigma + j\omega$*
1	2	$-3.451 \pm j3.141$
2	3	$-3.501 \pm j5.010$; -8.011
3	4	$-3.758 \pm j5.401$; $-9.8 \pm j5.672$
4	5	$-3.771 \pm j5.350$; $-10.40 \pm j12.38$; -13.78
5	6	$-3.770 \pm j5.349$; $-9.90 \pm j14.325$ -17.3; -22.077

* σ has the unit of sec^{-1}.
ω has the unit of rad/sec.

B. Convergence of the Residue of a Major Pole

The convergence of the residue of one of the major pole pairs (the residue of the second is the complex conjugate of the first) is illustrated in Table II.

Table II

RESIDUES OF THE MAJOR POLES

Degree of Approximation	Residue of the Major Poles	
	Magnitude	Phase Angle, deg
1	0.83	± 244.7
2	1.406	± 215.5
3	2.13	± 205
4	2.208	± 207.3
5	2.21	± 206.7

C. Convergence of the Initial and Final Values of Response for a Step Increase in Heat Flux

The initial and final values of $a_0(t)$ for the perturbed heat input specified before is given for the various approximations in Table III. The convergence is obvious. This completes the demonstration of convergence of a_0 in time.

Table III

ESTIMATION OF THE INITIAL AND FINAL VALUES

Degree of Approximation	Initial Value of a_0	Final Value of a_0
1	0.0382	0.1429
2	0.0391	0.2420
3	0.0386	0.2441
4	0.03856	0.2456
5	0.03855	0.2458

D. Convergence in Space

Convergence in space may be demonstrated only by showing that the space distributions of $\Delta\alpha$ and ΔV_f approach a particular shape at any particular instant since the spatial variation also changes with time. The magnitude of such a project does not make it feasible. Therefore, convergence will be demonstrated only for one point in space and time, namely, the location of the extremum of $\Delta\alpha$ after the transients due to a step increase in heat flux have decayed. This is shown in Table IV.

Table IV

LOCATION OF THE EXTREMUM OF $\Delta\alpha$ ALONG THE CHANNEL AFTER TRANSIENTS HAVE DECAYED

Degree of Approximation	Location of $\Delta\alpha$ Max-steady State, ft above the Inlet
1	1.0L
2	0.683L
3	0.623L
4	0.614L
5	0.616L

VI. APPLICATION TO SPECIFIC PROBLEMS

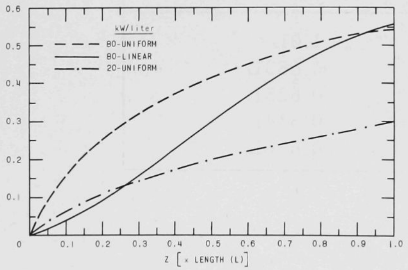
Several important questions connected with the dynamics of two-phase flow through heated boiling channels are now considered. In many instances, experimental data and digital-computer solutions are available to compare with the solutions predicted by the present analytical approach.

A. Dependence of Stability upon Steady-state Conditions

It is found for small perturbations that the stability and transient behavior of a single-channel natural-circulation loop depend upon the steady-state distributions of velocity and void fraction along the channel.[6,11] One may note that these in turn mainly depend upon the total heat flux of the channel and the form in which heat is distributed along the channel. For any given steady-state heat distribution the corresponding distributions of the void fraction $\alpha_0(z)$, velocity of water $V_{f0}(z)$, and the slip ratio $S(z)$ can be obtained using the digital computer program "CHOPPED."[12] The particular steady-state condition that will be considered is for a pressure of 600 psig and inlet water at saturation temperature.

The function $\alpha_0(z)$, $V_{f0}(z)$, and $S(z)$ were computed for the channel at 3 different steady-state heat fluxes, namely, 20 and 80 kW/liter (the internal volume of the channel) distributed uniformly, and 80 kW/liter distributed linearly (zero at the inlet). Figures 3a, b, and c illustrate the corresponding steady-state conditions. The coefficients of the Legendre polynomials used to fit the curves[10] in Figs. 3a, b, and c are given in Table V.

It is found that the solution converged reasonably well by the third-degree approximation in z. The locations of the major poles that dictate the stability and transient behavior corresponding to the three different steady-state conditions are indicated on Fig. 4.



112-3794

Fig. 3a. Variation of Void Fraction along a Channel

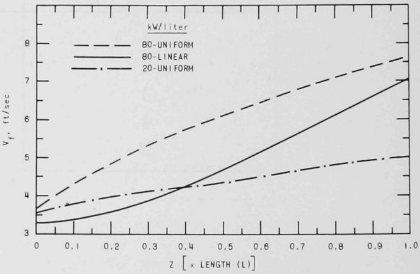


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Fig. 3b. Variation of Velocity of Water along a Channel

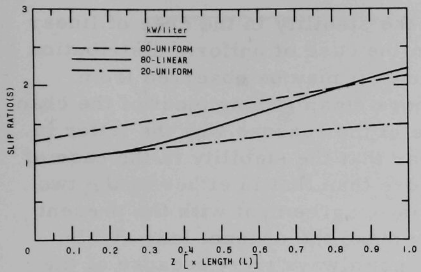


Fig. 3c

Variation of Slip Ratio along a Channel

112-3793

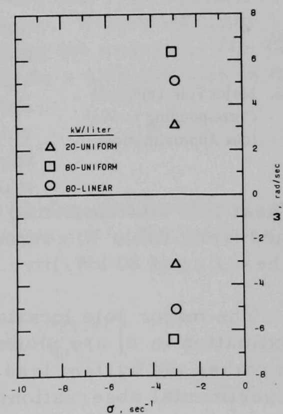
Table V

COEFFICIENTS OF LEGENDRE POLYNOMIALS

Variable	Coefficient	80 kW/liter, Uniform	80 kW/liter, Linear	20 kW/liter, Uniform
$\alpha_0(x)$ Fig. 3a	B_0 B_1 B_2	0.28 0.25 -0.10	0.29 0.29 0	0.18 0.18 0
$V_{f0}(x)$ Fig. 3b	A_0 A_1 A_2	6.01 2.02 -0.31	4.90 1.81 0	4.40 0.71 0
$S(x)$ Fig. 3c	S_0 S_1	1.80 0.50	1.6 0.6	1.25 0.10

Fig. 4

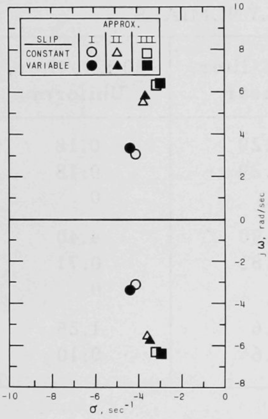
Variation of Major Pole Locations with Channel Heat Flux



112-3656

One may infer from Fig. 4 that the stability in the case of linear distribution of heat is more than that in the case of uniform distribution of the same 80 kW/liter of total heat flux. It may be observed from Fig. 3a that there is more void (i.e., more steam) along most of the channel in the latter case. This may be one of the reasons why the latter is less stable. From Fig. 4 it is also clear that the stability in the case of 20 kW/liter distributed uniformly is more than that in either of the two cases mentioned above. This, indeed, is in agreement with the present belief that increase in power input does make the channel less stable. However, it should be noted that this is not always true, because of the fact that the distribution of heat along the channel also influences stability.

It has been a usual assumption in past approaches that the slip ratio is a constant all along the channel. The present method takes the z-variation of the slip ratio into account. The major pole locations for the case of 80 kW/liter of heat distributed uniformly are plotted in Fig. 5, along with those obtained by neglecting just the z-variation of the slip ratio. One may readily observe from this figure that the error introduced in the prediction of stability with the assumption of a constant slip ratio is small for each approximation, provided the average value of the slip ratio is taken as the constant value in the mass and energy balance Eqs. (14) and (16).



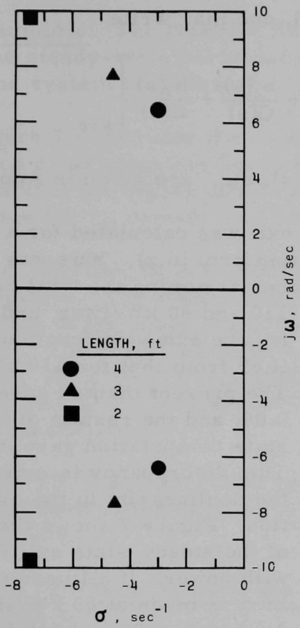
112-3655

Fig. 5. Major Pole Locations Corresponding to Various Approximations

B. Effect of Channel Length upon Dynamic Behavior

To study the effect of channel length upon dynamic behavior, three different lengths, namely, L = 2, 3, and 4 ft, of a natural-circulation loop were considered. It was assumed that the steady-state distributions of void fraction, velocity, and the slip ratio were identical in each case. (This, of course, implies that the channels have different steady-state heat flux distributions.) The values of A_0, A_1, B_0, B_1, etc., were obtained from Table V, corresponding to a channel, 4 ft long, with a uniform heat flux of 80 kW/liter.

The major pole locations for each of the above cases (third degree approximation in z) are plotted in Fig. 6. It is evident that an increase in length makes the system less stable. This conclusion is in agreement with the experimental observations of Meyer et al.[2]



112-3787

Fig. 6

Effect of Channel Length on Major Pole Locations

C. Prediction of the Steady State after the Transients

Example 1: The transfer function, $\Delta\alpha(x,s)/C(s)$ may be utilized to predict the steady value of the perturbation in the void fraction at any point along the channel after the transients have died out. The system considered here is the same as that in Part A of this section. Eighty kW/liter of heat were assumed to be distributed uniformly along the channel. The perturbation also was taken to be uniform in z and to be a step function in time. Therefore, $\Delta\phi$ had the form

$$\Delta\phi(x,s) = c_0(1/s).$$

The steady value of the perturbation in the void fraction at the exit may be obtained by applying the final value theorem to the transfer function $\Delta\alpha(1,s)/C(s)$ as given below:

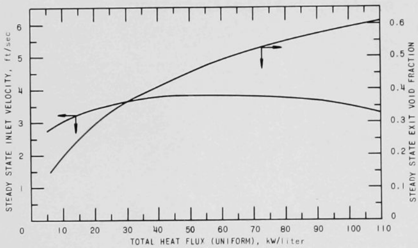
$$\Delta\alpha(1,t)\Big|_{t=\infty} = sC(s)\left(\frac{\Delta\alpha(1,s)}{C(s)}\right)\Bigg|_{s=0} = \frac{\Delta\alpha(1,s)}{C(s)}\Bigg|_{s=0} .$$

If a third-order approximation is used, one may write

$$\Delta\alpha(1,t)\Bigg|_{t=\infty} = \left[\frac{b_0(s)}{C(s)} + \frac{b_1(s)}{C(s)} + \frac{b_2(s)}{C(s)} + \frac{b_3(s)}{C(s)}\right]_{s=0} .$$

The transfer functions $b_0(s)/C(s)$, etc., are given in Appendix A.

For the steady state $\Delta\alpha$ at the exit was calculated for a perturbation of +30 kW/liter of total heat flux (uniform in z). This was also obtained from the steady-state computations assuming the total heat flux at 110 and 80 kW/liter, and subtracting the exit void fraction at 80 kW/liter from that for 110 kW/liter. The present method gave a value of 0.09, and the results of the steady-state computation gave 0.07 for $\Delta\alpha$. This discrepancy is mainly due to the nonlinearity in the exit void fraction. Figure 7 shows the variation of the steady-state exit void fraction with power. If a linear extrapolation in α is made at 80 kW/liter to obtain $\Delta\alpha$ between 80 and 110 kW/liter, one may readily see that the extrapolated result is bound to be more than the actual, considering the large size of the perturbation.



112-3797

Fig. 7. Variation of Exit Void Fraction and Inlet Velocity with Heat Flux

As the perturbation of the heat input in the above example was rather large, a second example was considered in which a smaller value for the perturbation, only 10 kW/liter, was taken. Perturbation in the exit void fraction was calculated for +10 and -10 kW/liter by the present method to obtain that between 70 and 90 kW/liter. This perturbation was also obtained from the steady-state information in Fig. 7. The values obtained for $\Delta\alpha$ were, respectively, 0.06 and 0.05.

At this point, one may note that the steady-state distributions of α, V_f, and S along the channel are expressed through the coefficients A_0, A_1, B_0, B_1, etc., obtained through a curve-fitting procedure as mentioned before. An exact fitting is impossible with a limited number of orthogonal polynomials, and some error therefore is introduced. Further, the steady-state computations are semi-empirical. Considering these facts, one may conclude that the small discrepancies in $\Delta\alpha$ in the above examples are quite reasonable.

Example 2: The transfer function $\Delta V_f(x,s)/C(s)$ may be used to evaluate the steady-state perturbation of the circulation velocity (inlet velocity). The system considered was the same as in Example 1.

Figure 7 illustrates the variation of the circulation velocity with power. Such a variation has been observed in many past experiments.(3,6) Consideration of the bigger nonlinearity of this curve led to the choice of a small value for c_0, namely, +10 kW/liter.

At 80 kW/liter of heat input, the curve is almost flat. The small slope is negative. The steady-state perturbation in the circulation velocity corresponding to the above perturbation in the input was found to be -0.04 ft/sec. The predicted value due to the transfer-function method was -0.01 ft/sec. Because the magnitude of the velocity perturbation at this power is so small, the inherent errors make any comparison between the two values meaningless.

At 20 kW/liter, the curve has a finite slope. Figure 7 gives the value of the perturbation at this power as +0.23 ft/sec; the predicted value was +0.25 ft/sec.

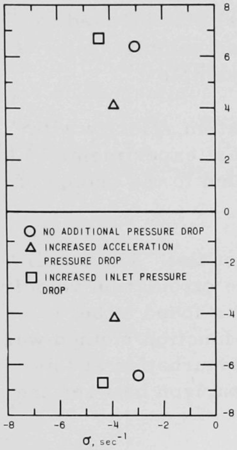
D. Effect of Pressure Drops in the Downcomer and at the Inlet to the Channel on Stability

The stability of the single-channel system described in Part A of this section, with a heat flux of 80 kW/liter distributed uniformly, was investigated with the following modifications.

1. The acceleration pressure drop in the downcomer was increased to 50 $s(a_0 - a_1 + a_2 - a_3)$. This may be done by suitably increasing its length or decreasing its diameter (see Appendix A).

2. The perturbed pressure drop at the inlet was increased to $200(a_0 - a_1 + a_2 \% a_3)$, by introducing a constriction before the inlet (see Appendix A).

Figure 8 gives the variation of the location of the major poles of the system due to the above additional pressure losses in the momentum-balance equation. It may be observed from this figure that presence of these pressure drops does increase the stability of the system. It has been observed in the past that increase in inlet drop leads to improvement of stability of steam boilers and test loops. The predicted effect of downcomer acceleration on stability is yet to be confirmed experimentally.



112-3788

Fig. 8. Major Pole Locations for the Effect of External Pressure Drops on Flow Stability

E. Parallel-channel System with a Common Downcomer

A system consisting of two heated channels, each 4 ft long and with a diameter of 1 in., operating in parallel with a common downcomer was considered. The downcomer was assumed to have variable length and diameter in order to permit variation of the acceleration pressure drop in it. A constriction may be introduced at the bottom of the downcomer in order to increase flow resistance. This system was studied in order to understand the effect of coupling between the flows in the two channels due to the presence of a common downcomer.

To obtain the transient behavior of α and V_f, one has to solve the conservation equations pertinent to both the channels simultaneously. The unknowns: a_0, a_1, b_0, b_1, etc., corresponding to both channels appear in the momentum-balance equations due to the common downcomer.

The system considered was in many respects similar to an RLC electrical network with two current paths having mutual impedance (mutual resistance and inductance).

It was assumed that the perturbations in the heat fluxes of the two channels were in phase. However, the z-variation of $\Delta\phi$ need not be the same along the two channels. The z-variation of ϕ_0 may also be different from that of $\Delta\phi$ along the channels. However, in the specific system considered, the z-variation of ϕ_0 and $\Delta\phi$ of each channel was assumed to be uniform. The total heat fluxes were taken to be 80 kW/liter for one channel and 20 kW/liter for the other in the steady state.

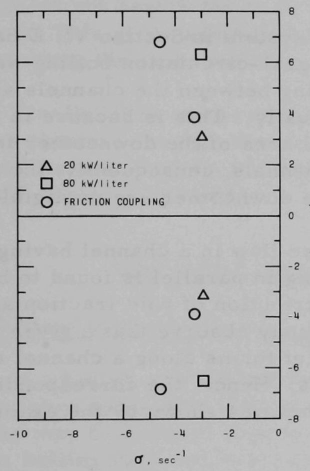
Figure 9 shows the locations of the major poles when the coupling was chosen to be small (corresponding to a downcomer length of 4 ft and diameter of 6 in.). These poles essentially coincide with those of the individual channels operated separately. One may say, therefore, that in the case of a boiling water reactor, if the downcomer and the perturbations of inlet pressure drop are small, the coolant channels may retain their individual characteristics insofar as the transient flow is concerned.

When the pressure drop due to acceleration in the downcomer was increased to $50\ s(a_0 - a_1 + a_2 - a_3 + a_0' - a_1' + a_2' - a_3')$* and that at the constriction

*Primed symbols correspond to the unknown coefficients of the second channel and the unprimed symbols to those of the first.

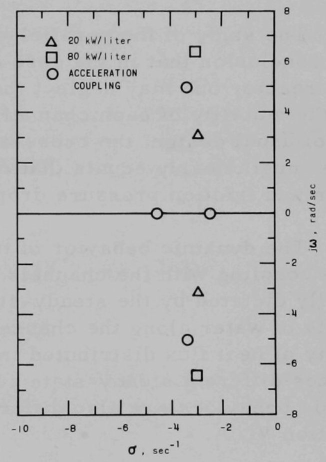
increased to make the total friction pressure drop (perturbed) in it equal to $200(a_0 - a_1 + a_2 - a_3 + a_0' - a_1' + a_2' - a_3')$, the coupling between the two channels was increased. The coupling due to the former pressure drop is called "acceleration coupling," that due to the latter "friction coupling." The reader is referred to Appendix A where the equations are shown in detail.

One may observe the shifting of the poles due to each of the above modifications in the downcomer pressure drop in Figs. 9 and 10. From this one may conclude that the dynamic behavior of the two channels are similar to that of the electrical network.



112-3796

Fig. 9. Major Pole Locations with and without Friction Coupling



112-3795

Fig. 10. Major Pole Locations with and without Acceleration Coupling

VII. DISCUSSION AND CONCLUSION

An important observation due to the present method of solution of the linearized perturbed form of the conservation equations is that during transients, the z-dependence of both void fraction and velocity is nonstationary. The transient void fraction undergoes damped spatial oscillations before its steady-state distribution along the channel is attained. This nature of the solution was also obtained by J. E. Meyer et al.,[2,4] as a result of solving the equations numerically. It was shown previously in Section IV that this conclusion is of special importance in boiling water nuclear reactors in that the spatial oscillations of the void fraction directly influence the transient void reactivity.

Figure 5 and Tables I-IV show that a sizable difference exists between the solution obtained by the first-degree approximation (z-dependence of $\Delta\alpha$ and ΔV_f in the first-degree approximation is linear) and those by the second- or higher-degree approximations. One may also observe that the discrepancy is a maximum between the first- and the second-degree approximations, and that the higher-degree approximations converge. This may be explained by the fact that in the first-degree approximation, the form of the solution is considerably restricted. Further, one may expect error in the time dependence of the solution when it is derived by forcing an erroneous preconceived form for its z-dependence.

The study of the parallel-channel system in Section VI, E has led to the conclusion that in the case of a natural-circulation boiling water power reactor one may neglect the coupling between the channels and treat the stability of each channel individually. This is because in a reactor of usual design, the cross-sectional area of the downcomer is large and its length nearly equals that of the channels; consequently, the acceleration and friction pressure drops in the downcomer are negligible.

The dynamic behavior of two-phase flow in a channel having negligible coupling with the channels operating in parallel is found to be essentially dictated by the steady-state distribution of void fraction and velocity of water along the channel. One may observe that a given total quantity of heat flux distributed in different forms along a channel establishes different steady-state conditions. Hence, the corresponding dynamic behaviors are also different. This was shown by the example in Section VI, A.

Figure 6 indicates that the length of the channel strongly influences the dynamic behavior. For the same steady-state distribution of void fraction and velocity, the oscillations in a shorter channel are at higher frequencies than those in a longer channel. The damping factor decreases with length; consequently, one may conclude that the longer channels are more likely to be unstable. An experimental confirmation of this fact may be found in Ref. 2.

It has been observed in the past that the introduction of a constriction at the inlet to a channel, in order to increase the inlet pressure drop, has a tendency to stabilize the flow. A theoretical verification of this was found in Section VI, D. Further, the present analysis shows that the acceleration pressure drop in the downcomer also stabilizes the flow. Increased acceleration pressure drop in the downcomer may be achieved by increasing its length or by reducing its cross-sectional area, or both. An experimental confirmation of this predicted effect of acceleration pressure drop in the downcomer is desirable.

It is noted once again that the present approximation procedure is applicable to both natural- and forced-circulation systems. If in the latter case a constant pressure drop exists between the inlet and exit of the channels, one of the boundary conditions would then be that the sum of the perturbed pressure drops at the inlet, along the channel, and at the outlet is equal to zero. In the analysis of a natural-circulation system, however, the perturbed pressure drop in the downcomer also has to be taken into account.

In the present analysis, the z-dependence of $\Delta\phi(z,t)$, the perturbed heat flux, is not restricted to be the same as that of $\phi_0(z)$, the steady-state heat flux. This gives additional flexibility to the design of an experimental setup to study the transients.

Finally, it is stressed that although the several results and conclusions obtained through the present analysis are valid only within the realm of the assumptions made in Section II, they are applicable in many practical cases of interest.

VIII. SUGGESTIONS FOR FUTURE WORK

A. A System in Which Subcooling is Present

In a boiler or a boiling water reactor, the makeup water introduced may be below saturation temperature. Therefore, the inlet water to the channels may be several degrees below saturation temperature. Consequently, boiling does not occur right from the inlet, and a portion of the length of the channel may have subcooled liquid.

The present analytical method cannot be applied directly in the presence of subcooling because the approximating expression for $\Delta\alpha$ becomes unrealistic. This is because α_0 and $\Delta\alpha$ are both equal to zero in the entire nonboiling portion of the channel. The series approximation of $\Delta\alpha$ is applicable in the boiling portion of the channel. However, L_b, the boiling length, becomes a function of time during transients and therefore would introduce another unknown in the equations.

It is suggested that the more general case including subcooling be treated analytically by writing the conservation equations involving a set of different unknowns, namely, the average density $\bar{\rho}(z,t)$, the mixing cup enthalpy of the fluid, $H(z,t)$, and the mass flow rate per unit flow area, $G(z,t)$. The functions $\alpha(z,t)$, $V_f(z,t)$, and $V_g(z,t)$ may be indirectly obtained from the solutions for the new unknowns.

The conservation equations which describe the flow in a channel may be written as follows (these equations are derived in detail by J. E. Meyer in Ref. 9):

Energy:

$$\rho \frac{\partial H}{\partial t} + G\frac{\partial H}{\partial z} = \phi ; \tag{57}$$

Mass:

$$\frac{\partial \rho}{\partial t} + \frac{\partial G}{\partial z} = 0; \tag{58}$$

Momentum:

$$\frac{\partial G}{\partial t} + \frac{\partial}{\partial z}(v' G^2) = -\frac{\partial P}{\partial z} - \frac{fG^2}{2D} - \bar{\rho} g. \tag{59}$$

In these equations:

1. Slip-flow effects are represented by use of the macroscopic quantities $\bar{\rho}$, v', and ρ'', which are assumed to be functions of mixing cup enthalpy H for a given pressure and are evaluated by steady-state experiments:

$$\rho'' = F_1(H); \tag{60}$$

$$\bar{\rho} = F_2(H); \tag{61}$$

$$v' = F_3(H). \tag{62}$$

2. All fluid properties are evaluated at some reference pressure.

Therefore, in Eqs. (57) to (59) the unknowns are only $G(z,t)$ and $H(z,t)$.

The linearized perturbed form of Eqs. (57) and (58) may be written after substituting for ρ'' and $\bar{\rho}$ from Eqs. (60) and (61). The perturbed form of Eq. (59) is integrated along the channel to obtain the perturbed pressure drop in the channel. The perturbed pressure drop in the rest of the flow loop may be added to this and the result equated to zero to obtain one of the two boundary conditions. If the inlet water is assumed to be at constant temperature (not a very restrictive assumption at all), one may obtain the second boundary condition: $H = 0$ at $z = 0$ at all times.

The advantage in making H and G the unknowns is that they would usually be continuous functions of z even when subcooling is present. If it is further assumed that

$$\Delta\phi = f(z)C(t),$$

where $f(z)$ is known, then, in principle, the present approximation procedure can be applied directly to solve for $\Delta H(z,t)$ and $\Delta G(z,t)$, given any Laplace transformable $C(t)$. The functions $\Delta\alpha(z,t)$, $\Delta\rho(z,t)$, etc., may then be obtained through ΔH and ΔG. Using $\Delta H(z,t)$, one may also derive the time variation of the boiling length during small transients.

B. Experimental Program

1. The effect of steady-state heat-flux distributions along the channel discussed in Section VI, A needs further experimental confirmation. This experiment may be facilitated by using the fact that in the present analysis the axial profiles of the steady-state and perturbed heat fluxes may be different.

2. The effect of acceleration pressure drop in the downcomer (increasing acceleration pressure drop is equivalent to increasing inertia to the fluid flow) has the effect of stabilizing the flow, as was shown in Section VI, D. This needs experimental confirmation.

3. Until now it has been virtually impossible to measure realistically the amount of steam in a channel during transients. With the development of the present method, it is only necessary to measure the value of α at any given location to verify the spatial distribution of $\alpha(z,t)$. Obtaining the total steam void is then a simple matter of integration of this function in z. Performing this experiment at various positions may be used to confirm the oscillatory nature of $\Delta\alpha(z,t)$ along the channel.

Appendix A

ILLUSTRATION OF THE TECHNIQUES IN THE APPROXIMATION PROCEDURE*

Single-channel System

A single heated channel with a downcomer is first considered. No coupling effect due to the other channels operating in parallel is assumed. After illustrating the analysis of this system, a system with two heated channels operating in parallel is considered to show how one could analyze if coupling exists between them due to the common downcomer.

Let the heated channel have a cross-sectional area A and length L. The area of cross section of the downcomer is taken to be A_d and the length L.

The linearized perturbed forms of the conservation equations of mass, energy, and momentum in the independent variables x and s are rewritten as follows:

$$s\frac{L}{2}(\rho_g-\rho_f)\Delta\alpha+(\rho_g S-\rho_f)\frac{\partial}{\partial x}[\alpha_0\Delta V_f+V_{f0}\Delta\alpha]$$

$$+\rho_f\frac{\partial\Delta V_f}{\partial x}+\rho_g\frac{dS}{dx}[V_{f0}\Delta\alpha+\alpha_0\Delta V_f]=0; \qquad \text{(A-1)}$$

$$s\frac{L}{2}(\rho_g h_g-\rho_f h_f)\Delta\alpha+(\rho_g h_g S-\rho_f h_f)\frac{\partial}{\partial x}[\alpha_0\Delta V_f+V_{f0}\Delta\alpha]$$

$$+\rho_f h_f\frac{\partial\Delta V_f}{\partial x}+\rho_g h_g\frac{dS}{dx}[V_{f0}\Delta\alpha+\alpha_0\Delta V_f]$$

$$=\frac{L}{2A}f(x)C(s); \qquad \text{(A-2)}$$

$$\frac{\partial\Delta P}{\partial x}=Lk_f V_{f0}\Delta V_f+\frac{L}{2}g(\rho_g-\rho_f)\Delta\alpha$$

$$+s\frac{L}{2}[(V_{f0}\Delta\alpha+\alpha_0\Delta V_f)(\rho_g S-\rho_f)+\rho_f\Delta V_f]$$

$$+\frac{\partial}{\partial x}[V_{f0}^2(\rho_g S^2-\rho_f)\Delta\alpha+2V_{f0}\alpha_0(\rho_g S^2-\rho_f)\Delta V_f+2\rho_f V_{f0}\Delta V_f].$$

$$\text{(A-3)}$$

C(s) is arbitrary.

*Appendix B contains some properties of Legendre polynomials which are used in the following.

Let ΔV_f and $\Delta\alpha$ be approximated to the third degree in x:

$$\Delta V_f = a_0 + a_1 P_1(x) + a_2 P_2(x) + a_3 P_3(x); \tag{A-4}$$

$$\Delta\alpha = b_0 + b_1 P_1(x) + b_2 P_2(x) + b_3 P_3(x). \tag{A-5}$$

The steady-state terms are assumed to be as follows:

$$V_{f0} = A_0 + A_1 P_1(x) + A_2 P_2(x); \tag{A-6}$$

$$\alpha_0 = B_0 + B_1 P_1(x) + B_2 P_2(x); \tag{A-7}$$

$$S = S_0 + S_1 P_1(x). \tag{A-8}$$

The space variation of the perturbation in heat is given by

$$f(x) = c_0 + c_1 P_1(x) + c_2 P_2(x). \tag{A-9}$$

Some new constants are defined below:

$$\rho_g - \rho_f = N_1; \tag{A-10}$$

$$\rho_g S_0 - \rho_f = N_4; \tag{A-11}$$

$$\rho_g h_g - \rho_f h_f = M_1; \tag{A-12}$$

$$\rho_g h_g S_0 - \rho_f h_f = M_4. \tag{A-13}$$

Equations (A-1) and (A-2) are rewritten after introducing the new constants defined in Eqs. (A-10) to (A-13), and substituting for S from Eq. (A-8):

$$s\frac{L}{2}\,\Delta\alpha + \frac{N_4}{N_1}\frac{\partial}{\partial x}(\alpha_0\,\Delta V_f + V_{f0}\,\Delta\alpha) + \frac{\rho_f}{N_1}\,\frac{\partial \Delta V_f}{\partial x}$$

$$+ \frac{\rho_g S_1}{N_1}(\alpha_0\,\Delta V_f + V_{f0}\,\Delta\alpha) + \frac{\rho_g S_1}{N_1}\,x\,\frac{\partial}{\partial x}(\alpha_0 \Delta V_f + V_{f0}\Delta\alpha) = 0; \tag{A-14}$$

$$s\frac{L}{2}\,\Delta\alpha + \frac{M_4}{M_1}\frac{\partial}{\partial x}(\alpha_0\,\Delta V_f + V_{f0}\,\Delta\alpha) + \frac{\rho_f h_f}{M_1}\,\frac{\partial \Delta V_f}{\partial x}$$

$$+ \frac{\rho_g h_g S_1}{M_1}(\alpha_0\,\Delta V_f + V_{f0}\,\Delta\alpha) + \frac{\rho_g h_g S_1}{M_1}\,x\,\frac{\partial}{\partial x}(\alpha_0\,\Delta V_f + V_{f0}\,\Delta\alpha) = \frac{L}{2A}\,f(x)C(s). \tag{A-15}$$

Subtracting Eq. (A-14) from (A-15),

$$\left(\frac{M_4}{M_1} - \frac{N_4}{N_1}\right)\frac{\partial}{\partial x}(\alpha_0\,\Delta V_f + V_{f0}\,\Delta\alpha) + \left(\frac{\rho_f h_f}{M_1} - \frac{\rho_f}{N_1}\right)\frac{\partial \Delta V_f}{\partial x} + \left(\frac{\rho_g h_g}{M_1} - \frac{\rho_g}{N_1}\right)S_1(\alpha_0\,\Delta V_f + V_{f0}\,\Delta\alpha)$$

$$+ \left(\frac{\rho_g h_g}{M_1} - \frac{\rho_g}{N_1}\right)S_1\, x\,\frac{\partial}{\partial x}(\alpha_0\,\Delta V_f + V_{f0}\,\Delta\alpha) = \frac{L}{2A}\, f(x)\, C(s). \tag{A-16}$$

Equations (A-14) and (A-16) are used as the mass and energy balance equations. Equation (A-3) is the momentum balance equation.

It is observed that there are 9 unknowns, including C(s). Six independent equations are obtained by multiplying each of the Eqs. (A-14) and (A-16) by $P_0(x)$, $P_1(x)$, and $P_2(x)$, respectively, and integrating over x between the limits -1 and +1 (orthogonality conditions). The two boundary conditions, namely, the perturbed pressure drop around a closed hydraulic loop is zero, and $\Delta\alpha = 0$ at $x = -1$ (that is, $z = 0$) yield two more independent equations. The eight equations thus obtained would enable one to evaluate the transfer functions $a_0(s)/C(s)$, $a_1(s)/C(s)$, etc.

The quantities $(\alpha_0\,\Delta V_f + V_{f0}\,\Delta\alpha)$, $\frac{\partial}{\partial x}(\alpha_0\,\Delta V_f + V_{f0}\,\Delta\alpha)$, and $\frac{\partial}{\partial x}\Delta V_f$ are expressed below by means of the substitutions corresponding to Eqs. (A-4) to (A-9). This is done to apply the orthogonality conditions to Eqs. (A-14) and (A-16) conveniently.

$$\begin{aligned}
\alpha_0\,\Delta V_f + V_{f0}\,\Delta\alpha = {} & (a_0B_0 + A_0b_0)P_0(x) + P_1(x)(a_0B_1 + B_0a_1 + A_0b_1 + b_0A_1) + P_2(x)(a_0B_2 + B_0a_2 + A_0b_2 + b_0A_2) \\
& + P_3(x)(B_0a_3 + A_0b_3) + \left[\frac{2}{5}P_1(x) + \frac{3}{5}P_3(x)\right](a_1B_2 + B_1a_2 + b_1A_2 + A_1b_2) \\
& + \left[\frac{3}{7}P_2(x) + \frac{4}{7}P_4(x)\right](B_1a_3 + A_1b_3) + \left[\frac{P_0(x)}{3} + \frac{2}{3}P_2(x)\right](a_1B_1 + b_1A_1) \\
& + \left[\frac{1}{5}P_0(x) + \frac{2}{7}P_2(x) + \frac{18}{35}P_4(x)\right](a_2B_2 + b_2A_2) \\
& + \left[\frac{9}{35}P_1(x) + \frac{4}{15}P_3(x) + \frac{10}{21}P_5(x)\right](B_2a_3 + A_2b_3);
\end{aligned} \tag{A-17}$$

$$\begin{aligned}
\frac{\partial}{\partial x}(\alpha_0\Delta V_f + V_{f0}\,\Delta\alpha) = {} & P_0(x)\,(a_0B_1 + B_0a_1 + A_0b_1 + b_0A_1) + 3P_1(x)\,(a_0B_2 + B_0a_2 + b_0A_2 \\
& + [P_0(x) + 5P_2(x)](B_0a_3 + A_0b_3) + [P_0(x) + 3P_2(x)](a_1B_2 + B_1a_2 + b_1A_2 + A_1b_2) \\
& + [3P_1(x) + 4P_3(x)](B_1a_3 + A_1b_3) + 2P_1(x)(a_1B_1 + b_1A_1) \\
& + \left[\frac{12}{5}P_1(x) + \frac{18}{5}P_3(x)\right](a_2B_2 + b_2A_2) + [P_0(x) + \frac{26}{7}P_2(x) + \frac{30}{7}P_4(x)](B_2a_3 + A_2b_3);
\end{aligned} \tag{A-18}$$

$$x \frac{\partial}{\partial x}(\alpha_0 \Delta V_f + V_{f0} \Delta\alpha) = P_1(x)(a_0B_1 + B_0a_1 + A_0b_1 + b_0A_1) + [P_0(x) + 2P_2(x)](a_0B_2 + B_0a_2 + A_0b_2 + b_0A_2)$$

$$+ [3P_1(x) + 3P_3(x)](B_0a_3 + A_0b_3) + \left[\frac{11}{5} P_1(x) + \frac{9}{5} P_3(x)\right](a_1B_2 + B_1a_2 + b_1A_2 + A_1b_2)$$

$$+ \left[P_0(x) + \frac{26}{7} P_2(x) + \frac{16}{7} P_4(x)\right](B_1a_3 + A_1b_3) + \left[\frac{2}{3} P_0(x) + \frac{4}{3} P_2(x)\right](a_1B_1 + b_1A_1)$$

$$+ \left[\frac{4}{5} P_0(x) + \frac{22}{7} P_2(x) + \frac{72}{35} P_4(x)\right](a_2B_2 + b_2A_2)$$

$$+ \left[\frac{174}{70} P_1(x) + \frac{62}{15} P_3(x) + \frac{150}{63} P_5(x)\right](B_2a_3 + A_2b_3); \tag{A-19}$$

$$\frac{\partial}{\partial x} \Delta V_f = P_0(x)a_1 + 3P_1(x)a_2 + [P_0(x) + 5P_2(x)]a_3. \tag{A-20}$$

Multiplying Eqs. (A-14) and (A-16) by $P_0(x)$, integrating in x between the limits -1 and +1, and dividing the resulting equations by 2, these are obtained

$$s\frac{L}{2} b_0 + \frac{N_4}{N_1}(a_0B_1 + B_0a_1 + A_0b_1 + b_0A_1 + B_0a_3 + A_0b_3 + A_1B_2 + B_1a_2 + b_1A_2 + A_1b_2 + B_2a_3 + A_2b_3) + \frac{\rho_f}{N_1}(a_1 + a_3)$$

$$+ \frac{\rho_g S_1}{N_1}\left[a_0B_0 + A_0b_0 + \frac{1}{3}(a_1B_1 + b_1A_1) + \frac{1}{5}(a_2B_2 + b_2A_2)\right]$$

$$+ \frac{\rho_g S_1}{N_1}\left[a_0B_2 + B_0a_2 + A_0b_2 + b_0A_2 + B_1a_3 + A_1b_3 + \frac{2}{3}(a_1B_1 + b_1A_1) + \frac{4}{5}(a_2B_2 + b_2A_2)\right] = 0; \tag{A-21}$$

$$\left(\frac{M_4}{M_1} - \frac{N_4}{N_1}\right)(a_0B_1 + B_0a_1 + A_0b_1 + b_0A_1 + B_0a_3 + A_0b_3 + a_1B_2 + B_1a_2 + b_1A_2 + A_1b_2 + B_2a_3 + A_2b_3)$$

$$+ \left(\frac{\rho_f h_f}{M_1} - \frac{\rho_f}{N_1}\right)(a_1 + a_3) + \left(\frac{\rho_g h_g}{M_1} - \frac{\rho_g}{N_1}\right) S_1 \left[a_0B_0 + A_0b_0 + \frac{1}{3}(a_1B_1 + b_1A_1)\right.$$

$$+ \frac{1}{5} a_2B_2 + b_2A_2 + (a_0B_2 + B_0a_2 + A_0b_2 + b_0A_2 + B_1a_3 + A_1b_3)$$

$$\left. + \frac{2}{3}(a_1B_1 + b_1A_1) + \frac{4}{5}(a_2B_2 + b_2A_2)\right] = \frac{Lc_0}{2A} C(s). \tag{A-22}$$

Multiplying Eqs. (A-14) and (A-16) by $P_1(x)$, integrating in x between the limits -1 and +1, and dividing the resulting equations by 2/3, we obtain

$$s\frac{L}{2} b_1 + \frac{N_4}{N_1}\left[3(a_0B_2 + B_0a_2 + A_0b_2 + b_0A_2 + B_1a_3 + A_1b_3) + 2(a_1B_1 + b_1A_1) + \frac{12}{5}(a_2B_2 + b_2A_2)\right] + \frac{\rho_f}{N_1}(3a_2)$$

$$+ \frac{\rho_g S_1}{N_1}\left[a_0B_1 + B_0a_1 + A_0b_1 + b_0A_1 + \frac{2}{5}(a_1B_2 + B_1a_2 + b_1A_2 + A_1b_2) + \frac{9}{35}(B_2a_3 + A_2b_3)\right]$$

$$+ \frac{\rho_g S_1}{N_1}\left[a_0B_1 + B_0a_1 + A_0b_1 + b_0A_1 + 3(B_0a_3 + A_0b_3)\right.$$

$$\left. + \frac{11}{5}(a_1B_2 + B_1a_2 + b_1A_2 + A_1b_2) + \frac{170}{70}(B_2a_3 + A_2b_3)\right] = 0; \tag{A-23}$$

$$\left(\frac{M_4}{M_1} - \frac{N_4}{N_1}\right)\left[3(a_0B_2 + B_0a_2 + A_0b_2 + b_0A_2 + B_1a_3 + A_1b_3) + 2(a_1B_1 + b_1A_1) + \frac{12}{5}(a_2B_2 + b_2A_2)\right] + \left(\frac{\rho_f h_f}{M_1} - \frac{\rho_f}{N_1}\right)(3a_2)$$

$$+ \left(\frac{\rho_g h_g}{M_1} - \frac{\rho_g}{N_1}\right)S_1\left[a_0B_1 + B_0a_1 + A_0b_1 + b_0A_1 + \frac{2}{5}(a_1B_2 + B_1a_2 + b_1A_2 + A_1b_2) + \frac{9}{35}(B_2a_3 + A_2b_3)\right]$$

$$+ \left(\frac{\rho_g h_g}{M_1} - \frac{\rho_g}{N_1}\right)S_1\left[a_0B_1 + B_0a_1 + A_0b_1 + b_0A_1 + 3(B_0a_3 + A_0b_3) + \frac{11}{5}(a_1B_2 + B_1a_2 + b_1A_2 + A_1b_2)\right.$$

$$\left. + \frac{174}{70}(B_2a_3 + A_2b_3)\right] = \frac{Lc_1}{2A}C(s). \qquad \text{(A-24)}$$

Multiplying Eqs. (A-14) and (A-16) by $P_2(x)$, integrating between the limits -1 and +1, and dividing the resulting equations by 2/5, we find

$$s\frac{L}{2}b_2 + \frac{N_4}{N_1}\left[5(B_0a_3 + A_0b_3) + \frac{26}{7}(B_2a_3 + A_2b_3) + 3(a_1B_2 + B_1a_2 + b_1A_2 + A_1b_2)\right] + \frac{\rho_f}{N_1}(5a_3)$$

$$+ \frac{\rho_g S_1}{N_1}\left[a_0B_2 + B_0a_2 + A_0b_2 + b_0A_2 + \frac{3}{7}(B_1a_3 + A_1b_3) + \frac{2}{3}(a_1B_1 + b_1A_1) + \frac{2}{7}(a_2B_2 + b_2A_2)\right]$$

$$+ \frac{\rho_g S_1}{N_1}\left[2(a_0B_2 + B_0a_2 + A_0b_2 + b_0A_2) + \frac{26}{7}(B_1a_3 + A_1b_3) + \frac{4}{3}(a_1B_1 + b_1A_1) + \frac{22}{7}(a_2B_2 + b_2A_2)\right] = 0$$

and (A-25)

$$\left(\frac{M_4}{M_1} - \frac{N_4}{N_1}\right)\left[5(B_0a_3 + A_0b_3) + \frac{26}{7}(B_2a_3 + A_2b_3) + 3(a_1B_2 + B_1a_2 + b_1A_2 + A_1b_2)\right] + \left(\frac{\rho_f h_f}{M_1} - \frac{\rho_f}{N_1}\right)(5a_3)$$

$$+ \left(\frac{\rho_g h_g}{M_1} - \frac{\rho_g}{N_1}\right)S_1\left[a_0B_2 + B_0a_2 + A_0b_2 + b_0A_2 + \frac{3}{7}(B_1a_3 + A_1b_3) = \frac{2}{3}(a_1B_1 + b_1A_1) + \frac{2}{7}(a_2B_2 + b_2A_2)\right]$$

$$+ \left(\frac{\rho_g h_g}{M_1} - \frac{\rho_g}{N_1}\right)S_1\left[2(A_1B_2 + B_0a_2 + A_0b_2 + b_0A_2) + \frac{26}{7}(B_1a_3 + A_1b_3) + \frac{4}{3}(a_1B_1 + b_1A_1)\right.$$

$$\left. + \frac{22}{7}(a_2B_2 + b_2A_2)\right] = \frac{Lc_2}{2A}C(s). \qquad \text{(A-26)}$$

The boundary condition that $\Delta\alpha = 0$ at $x = -1$ may be expressed as

$$b_0 - b_1 + b_2 - b_3 = 0. \qquad \text{(A-27)}$$

The second boundary condition is that the perturbed pressure drop around a closed hydraulic loop equals (perturbed pressure drop in the channel) + (that at the exit) + (that in the downcomer) + (that at the inlet) is 0.

The perturbed pressure drop in the channel is obtained by substituting Eqs. (A-4) to (A-13) into Eq. (A-3) and integrating in x between the limits -1 and +1:

$$
\begin{aligned}
-\Delta P_{\text{channel}} = {} & 2LK_f\left(a_0A_0 + \frac{a_1A_1}{3} + \frac{a_2A_2}{5}\right) + Lg(\rho_g - \rho_f)b_0 + sL\Big[N_4\Big\{(a_0B_0 + A_0b_0) \\
& + \frac{1}{3}(a_1B_1 + b_1A_1) + \frac{1}{5}(a_2B_2 + b_2A_2)\Big\} + \rho_g a_0\Big] + sL\rho_g S_1\Big[\frac{1}{3}(a_0B_1 + B_0a_1 + A_0b_1 + b_0A_1) \\
& + \frac{2}{15}(a_1B_2 + B_1a_2 + b_1A_2 + A_1b_2) + \frac{3}{35}(B_2a_3 + A_2b_3)\Big] + (b_0 + b_1 + b_2 + b_3)\Big[V_{f0}^2(\rho_g S^2 - \rho_f)\Big]_{x=1} \\
& + (a_0 + a_1 + a_2 + a_3)\Big[V_{f0}\alpha_0(\rho_g S^2 - \rho_f) + \rho_f V_{f0}\Big]_{x=1} - (a_0 - a_1 + a_2 - a_3)(\rho_f V_{f0})_{x=1}.
\end{aligned}
\tag{A-28}
$$

The perturbed pressure drop at the exit is neglected along with the eddy momentum loss at the exit in accordance with the reasonings given in Section II.

The perturbed pressure drop in the downcomer is due to friction and acceleration in it, and is given by

$$
-\Delta P_D(s) = K_{f_D}\,\Delta V_D + K_{a_D}\,s\,\Delta V_D, \tag{A-29}
$$

where K_{f_D} and K_{a_D} are obtained from steady-state information, and ΔV_D is the perturbed velocity in the downcomer. The assumption of no steam carryunder in the downcomer is used in writing Eq. (A-29). Taking water to be incompressible, one may express continuity of flow through the downcomer and the channel as follows:

$$
\begin{aligned}
\Delta V_D &= \frac{A}{A_D}(\Delta V_f)_{x=-1} \\
&= \frac{A}{A_D}(a_0 - a_1 + a_2 - a_3).
\end{aligned}
\tag{A-30}
$$

Hence, Eq. (A-29) may be rewritten as

$$
-\Delta P_D(s) = K_{f_D}\frac{A}{A_D}(a_0 - a_1 + a_2 - a_3) + sK_{a_D}\frac{A}{A_D}(a_0 - a_1 + a_2 - a_3). \tag{A-31}
$$

The perturbed pressure drop at the inlet may be stated as

$$
\begin{aligned}
-\Delta P_{\text{inlet}} &= K_{\text{inlet}}(\Delta V_f)_{x=-1} \\
&= K_{\text{inlet}}(a_0 - a_1 + a_2 - a_3),
\end{aligned}
\tag{A-32}
$$

where K_{inlet} is computed from the steady-state data. Equations (A-28), (A-31), and (A-32) are added to obtain an expression for the perturbed pressure drop around the loop, which is equated to zero to obtain

$$
\begin{aligned}
&2LK_f \left(a_0A_0 + \frac{a_1A_1}{5} + \frac{a_2A_2}{5}\right) + Lg(\rho_f - \rho_g)b_0 + sL\Big[N_4\Big\{(a_0B_0 + A_0b_0) + \frac{1}{3}(a_1B_1 + b_1A_1) \\
&+ \frac{1}{5}(a_2B_2 + A_2b_2)\Big\} + \rho_g a_0\Big] + sL\rho_g S_1 \Big[\frac{1}{3}(a_0B_1 + B_0a_1 + A_0b_1 + b_0A_1) + \frac{2}{15}(a_1B_2 + B_1a_2 + b_1A_2 + A_1b_2) \\
&+ \frac{3}{35}(B_2a_3 + A_2b_3)\Big] + (b_0 + b_1 + b_2 + b_3)\Big[V_{f0}^2(\rho_g S^2 - \rho_f)\Big]_{x=1} + (a_0 + a_1 + a_2 + a_3)\Big[V_{f0}\alpha_0(\rho_g S^2 - \rho_f) \\
&+ \rho_f V_{f0}\Big]_{x=1} - (a_0 - a_1 + a_2 - a_3)\Big[\rho_f V_{f0}\Big]_{x=1} + \left(K_{f_D}\frac{A}{A_D} + K_{inlet} + sK_{a_D}\frac{A}{A_D}\right)(a_0 - a_1 + a_2 - a_3) = 0.
\end{aligned}
\tag{A-33}
$$

Equations (A-21) to (A-33) may be expressed in the following matrix equation form:

$$[A(s)][x(s)] = [B]C(s), \tag{A-34}$$

where $[A(s)]$ is an 8/8 matrix whose elements may contain s, $[x(s)]$ is a column matrix formed by the unknowns a_0, a_1, b_0, b_1, etc., and $[B]$ is another column matrix whose elements are the coefficients of $C(s)$ in Eqs. (A-21) to (A-33).

Therefore,

$$[x(s)] = [A(s)]^{-1}[B]. \tag{A-34a}$$

The transfer functions $a_0(s)/C(s)$, $a_1(s)/C(s)$, $b_0(s)/C(s)$, $b_1(s)/C(s)$, etc., are readily obtained from Eq. (A-34a). The operation $[A(s)]^{-1}[B]$ is performed by means of a digital computer program due to Guppy.[13]

<u>Example:</u> Consider a single-channel loop, 4 ft long and with a diameter of 1 in. The downcomer length is taken to be 4 ft and the diameter 6 in.

The values of A_0, A_1, B_0, B_1, etc., (corresponding to 80 kW/liter - uniform heat flux) are substituted in Eqs. (A-21) to (A-33). The resulting equations are arranged in the form of Eq. (A-34). This is shown in Fig. A-1. We take $f(x) = c_0$ and $(L/2A)c_0$ equal to 1 for convenience. By means of Guppy's digital computer program, the common denominator of the transfer functions $a_0(s)/C(s)$, $a_1(s)/C(s)$, etc., along with the respective numerators, are obtained.

Common Denominator:

$$
\begin{aligned}
&1.5585 \times 10^5\ s^4 + 4.8221 \times 10^6\ s^3 \\
&\quad + 6.1761 \times 10^7\ s^2 + 3.8168 \times 10^8\ s \\
&\quad + 1.5007 \times 10^9.
\end{aligned}
$$

$$
\begin{bmatrix}
0.680 & 0 & 2.698 & 20.010 & 0.680 & 0 & 2.698 & 20.010 \\
0 & 1.000 & 0 & -1.000 & 0 & 1.000 & 0 & -1.000 \\
0.021 & 0 & 1.300 & 20.178 & 8.089 & 30.273 & 2.065 & 29.336 \\
-0.300 & 0 & 0.021 & 1.661 & 2.000 & 27.366 & 13.480 & 53.580 \\
0.246 & \begin{matrix}2.046S\\+7.590\end{matrix} & -0.777 & 0 & 0.246 & 7.590 & -0.777 & 0 \\
-0.307 & 0 & 0.493 & \begin{matrix}2.046S\\+2.883\end{matrix} & -2.269 & 18.163 & 0.738 & 4.798 \\
0.004 & 0 & -0.307 & -0.943 & 0.739 & \begin{matrix}2.046S\\+5.746\end{matrix} & -3.750 & 28.785 \\
\begin{matrix}63.338S\\+125.820\end{matrix} & 0 & \begin{matrix}-7.692S\\+402.566\end{matrix} & \begin{matrix}-624.250S\\-5615.170\end{matrix} & \begin{matrix}1.928S\\+24.020\end{matrix} & \begin{matrix}570.100S\\+3101.560\end{matrix} & \begin{matrix}-0.011S\\+391.360\end{matrix} & \begin{matrix}-564.130S\\-5615.170\end{matrix}
\end{bmatrix}
\begin{bmatrix} a_0 \\ b_0 \\ a_1 \\ b_1 \\ a_2 \\ b_2 \\ a_3 \\ b_3 \end{bmatrix}
=
\begin{bmatrix} \frac{L}{2A} c_0 \\ 0 \\ \frac{L}{2A} c_1 \\ \frac{L}{2A} c_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} C(S)
$$

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Fig. A-1. Matrix Equation for Transfer Functions at 80 kW/liter - Uniform

Numerator of $a_0(s)/C(s)$:

$7.1972 \times 10^3\ s^4 + 2.9894 \times 10^4\ s^3 - 6.1898 \times 10^5\ s^2 - 3.3413 \times 10^7\ s + 3.6739 \times 10^8$.

Numerator of $a_1(s)/C(s)$:

$5.6870 \times 10^4\ s^4 + 1.8228 \times 10^6\ s^3 + 2.6843 \times 10^7\ s^2 + 1.2361 \times 10^8\ s + 3.0569 \times 10^8$.

Numerator of $a_2(s)/C(s)$:

$-9.5380 \times 10^3\ s^4 - 4.5770 \times 10^5\ s^3 - 9.6600 \times 10^5\ s^2 - 6.6697 \times 10^6\ s - 6.2960 \times 10^7$.

Numerator of $a_3(s)/C(s)$:

$1.4867 \times 10^3\ s^4 - 2.0311 \times 10^5\ s^3 - 1.2276 \times 10^6\ s^2 - 7.6197 \times 10^5\ s + 8.5000 \times 10^6$.

Numerator of $b_0(s)/C(s)$:

$2.2440 \times 10^4\ s^3 + 5.2870 \times 10^5\ s^2 + 1.1080 \times 10^7\ s + 3.7102 \times 10^7$.

Numerator of $b_1(s)/C(s)$:

$-2.3735 \times 10^4\ s^3 - 1.1088 \times 10^6\ s^2 - 1.4726 \times 10^4\ s + 1.2978 \times 10^7$.

Numerator of $b_2(s)/C(s)$:

$1.4692 \times 10^4\ s^3 - 8.4213 \times 10^5\ s^2 - 7.2076 \times 10^6\ s - 1.4812 \times 10^7$.

Numerator of $b_3(s)/C(s)$:

$6.0867 \times 10^4\ s^3 + 7.9537 \times 10^5\ s^2 + 3.8700 \times 10^6\ s + 9.3120 \times 10^6$.

Parallel-channel System

A system of two heated channels operating in parallel with a common downcomer may be analyzed by a method analogous to that for a system with only one heated channel. It is assumed that the perturbed heat input into the channels may be expressed as

$$\Delta\phi = f(z)C(s) \tag{A-35}$$

and

$$\Delta\phi' = f'(z)C(s), \tag{A-35a}$$

where f(z) and f'(z) are known, and C(s) is common for both Eqs. (A-35) and (A-35a). Primed symbols correspond to the second channel.

In a third-degree approximation of this system, there would be 17 unknowns: 8 for the first channel, 8 for the second channel, and 1 due to C(s). The 16 equations necessary to obtain the transfer functions would consist of 2 sets of 8 equations each. These sets would have a form identical to that of the system with a single heated channel, except that the terms corresponding to the perturbed pressure drop in the downcomer would contain the unknowns of both channels. One may express the perturbed pressure drop in the downcomer as

$$-\Delta P_D = \left(K_{f_D} + sK_{a_D}\right)\frac{A}{A_D}(a_0 - a_1 + a_2 - a_3)$$

$$+ \left(K_{f_D} + sK_a\right)\frac{A'}{A_D}(a_0' - a_1' + a_2' - a_3'). \tag{A-36}$$

<u>Example:</u> Consider a system of 2 channels working in parallel with a common downcomer. Each channel is 4 ft long and has a diameter of 1 in. The downcomer dimensions are variable.

The 16 equations are written in the form of Eq. (A-34), as is shown in Fig. A-2. The channels have a uniform steady-state heat flux at 80 kW/liter and 20 kW/liter, respectively. Table V gives the coefficients A_0, A_1, B_0, B_1, etc., corresponding to these heat fluxes. The downcomer has negligible friction and the perturbed acceleration pressure drop in it is taken as $50s[a_0 - a_1 + a_2 - a_3 + a_0' - a_1' + a_2' - a_3']$. The transfer functions may be readily evaluated if f(x) in each channel is specified.

$$
\begin{bmatrix}
0.680 & 0 & 2.698 & 20.010 & 0.680 & 0 & 2.698 & 20.010 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.000 & 0 & -1.000 & 0 & 1.000 & 0 & -1.000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.021 & 0 & 1.300 & 20.178 & 8.089 & 30.273 & 2.065 & 29.336 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.300 & 0 & 0.021 & 1.661 & 2.000 & 27.366 & 13.480 & 53.580 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.246 & 2.046S+7.590 & -0.777 & 0 & 0.246 & 7.590 & -0.777 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.307 & 0 & 0.493 & 2.046S+2.883 & -2.269 & 18.163 & 0.738 & 4.798 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.004 & 0 & -0.307 & -0.943 & 0.739 & 2.046S+5.746 & -3.750 & 28.785 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
113.337S+125.820 & 0 & -57.692S+402.566 & -624.250S-5615.170 & 51.928S+24.020 & 570.100S+3101.560 & -50.011S+391.360 & -564.130S-5615.170 & 0 & -50.000S & 0 & 50.000S & 0 & -50.000S & 0 & 50.000S \\
50.000S & 0 & -50.000S & 0 & 50.000S & 0 & -50.000S & 0 & -1162.000 & -50.000+356.000 & 0 & +50.000S-0.400 & -22.320S-1162.000 & -55.740S+360.200 & -420.860S-3101.560 & 131.780S+78.800 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 22.000 & -4.275 & 2.014S+2.100 & 0.540 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.100 & 0.540 & 13.200 & -2.565 & 2.014S+1.400 & 0.360 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4.400 & -0.855 & 0.700 & 0.180 & 4.400 & -0.855 & 2.014S+0.700 & 0.180 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 22.000 & 20.900 & 2.100 & 0.540 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.100 & 0.540 & 13.200 & 12.540 & 1.400 & 0.360 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.000 & 0 & 1.000 & 0 & -1.000 & 0 & 1.000 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4.400 & 4.180 & 0.700 & 0.180 & 4.400 & 4.180 & 0.700 & 0.180
\end{bmatrix}
\begin{bmatrix}
a_0 \\ b_0 \\ a_1 \\ b_1 \\ a_2 \\ b_2 \\ a_3 \\ b_3 \\ b_3' \\ a_3' \\ b_2' \\ a_2' \\ b_1' \\ a_1' \\ b_0' \\ a_0'
\end{bmatrix}
=
\begin{bmatrix}
\frac{L}{2A} c_0 \\ 0 \\ \frac{L}{2A} c_1 \\ \frac{L}{2A} c_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{L'}{2A'} c_2' \\ \frac{L'}{2A'} c_1' \\ 0 \\ \frac{L'}{2A'} c_0'
\end{bmatrix}
C(S)
$$

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Fig. A-2. Matrix Equation with Acceleration Coupling

Appendix B

LEGENDRE POLYNOMIALS

Legendre polynomials of the nth degree, $P_n(x)$, may be defined by the equation

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

for n = 0, 1, 2, 3, etc.

The polynomials of the lowest degrees are

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5).$$

The orthogonality of the Legendre polynomials is indicated by the equations

$$\int_{-1}^{+1} P_n(x)P_m(x)\, dx = 0 \text{ for } n \neq m;$$

$$\int_{-1}^{+1} [P_n(x)]^2\, dx = \frac{2}{2n+1}.$$

The asymptotic behavior of the Legendre polynomials may be described by the following equations:

$$P_n(1) = 1$$

$$P_n(-1) = (-1)^n$$

$$P_{2n+1}(0) = 0$$

$$P_2(0) = (-1)^n \frac{1.3 \ldots\ldots (2n-1)}{2.4 \ldots 2n} \text{ (not for } n = 0).$$

The term x^n may be expressed as a sum of the Legendre polynomials of degree up to and including n. For example:

$$1 = P_0$$

$$x = P_1$$

$$x^2 = \frac{1}{3}(P_0 + 2\ P_2)$$

$$x^3 = \frac{1}{5}(3\ P_1 + 2\ P_3)$$

$$x^4 = \frac{1}{35}(7\ P_0 + 20\ P_2 + 8\ P_4)$$

$$x^5 = \frac{1}{63}(27\ P_1 + 28\ P_3 + 8\ P_5)$$

$$x^6 = \frac{1}{231}(33\ P_0 + 110\ P_2 + 72\ P_4 + 16\ P_6).$$

More information about Legendre polynomials is available in Ref. 14.

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